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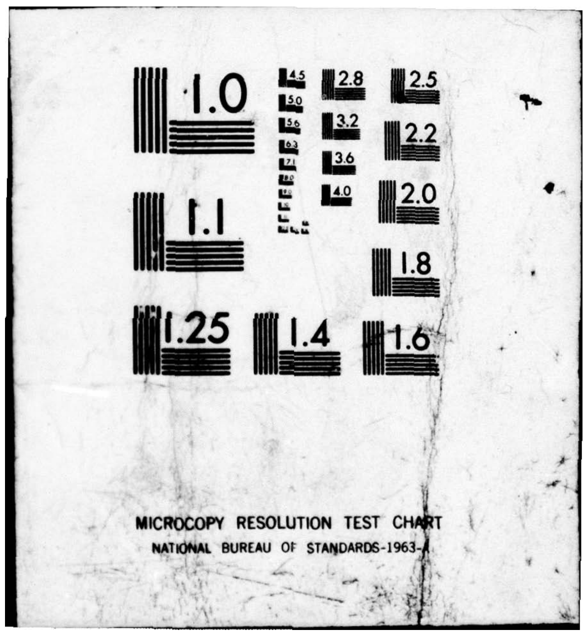
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# DETERMINING THE CHROMATIC NUMBER OF A GRAPH

by

Colin McDiarmid

STAN-CS-76-576
OCTOBER 1976

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
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Certain algorithms concerning coloring graphs involve the partial exploration of Zykov trees. We investigate the size of such trees, and prove that a certain class of branch-and-bound algorithms for determining the chromatic number of a graph requires in probability a number of steps which grows faster than exponentially with the number of vertices of the graph.

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Determining the Chromatic Number of a Graph

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Abstract

Certain algorithms concerning coloring graphs involve the partial exploration of Zykov trees. We investigate the size of such trees, and prove that a certain class of branch-and-bound algorithms for determining the chromatic number of a graph requires in probability a number of steps which grows faster than exponentially with the number of vertices of the graph.



*/ Visiting from Corpus Christi College, Oxford, England.

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Contents

## 1. Introduction.

Graph coloring problems arise in many practical situations, for example in various timetabling and scheduling problems (see for example [13], [14]). It would be very useful to be able to determine quickly the chromatic number of a graph. However, it is well known that this problem is NP-complete, and thus we do not expect to find good algorithms for the problem ([1], [10]). There has been proposed a class of branch-and-bound algorithms, which we call here Zykov algorithms (see [5]). We branch on whether or not two non-adjacent vertices have the same color and bound by using the fact that the chromatic number of a graph is at least the size of any complete subgraph. Zykov algorithms always explore at least a 'pruned Zykov tree'. We shall prove in Section 5 below that for almost all graphs $G_n$ on $n$ vertices every pruned Zykov tree has at least

$$c^{n(\log n)^{1/2}}$$

nodes, for some constant $c > 1$. It follows that any Zykov algorithm requires in probability more than exponential time.

E. L. Lawler [11] has recently noted that a simple algorithm involving the maximal stable sets of a graph requires only exponential time. This algorithm is then faster than the Zykov algorithms.

In the next section we give some preliminary definitions, including those of Zykov trees and Zykov algorithms, and in the following section we present some preliminary lemmas. After that, in Section 4 we investigate the size of Zykov trees. The standard algorithm for determining the chromatic polynomial of a graph involves the exploration of a Zykov tree (see for example [2] Chapter 15). In Section 5 we investigate the size

of pruned Zykov trees and deduce that Zykov algorithms are slow. We also give a numerical example.

In Section 6 we investigate a backtrack coloring algorithm. We show that it is essentially the same as a certain Zykov algorithm, and obtain an upper bound for the time it requires. Then in Section 7 we give an interpretation of our earlier results in terms of the lengths of certain proofs concerning the chromatic number. The results in this section are similar in spirit to some recent results of V. Chvatal [4] concerning stability numbers of graphs; and indeed the research reported in this paper was initially motivated by discussions with Chvatal concerning his results. Finally in Section 8 we consider 'minimal' coloring algorithms, which may use more colors than necessary, and investigate the ratio of the number of colors used to the chromatic number. This last section is not closely related in content to the rest of the paper, but the results there follow easily from lemmas used earlier.

## 3. Preliminary Definitions.

A proper coloring of a graph $G$ (without loops or parallel edges) is a coloring of the vertices of $G$ so that no two adjacent vertices receive the same color. The color sets in such a coloring form a proper partition of $G$. The chromatic number $\chi(G)$ is the least integer $k$ such that there is a proper coloring of $G$ using $k$ colors. A graph is complete if every two vertices are adjacent, and the clique number $\omega(G)$ is the greatest number of vertices in a complete subgraph of $G$.

Let $n$ be a positive integer. We let $\mathcal{G}_n$ denote the set of all graphs with vertex set $\{1,\ldots,n\}$. Throughout the paper $p$ will be a constant with $0 < p < 1$ and $q$ will be $1-p$. A probability distribution is induced on the set $\mathcal{G}_n$ of graphs by the statement that each edge occurs independently with probability $p$. If $k$ is a positive integer and $0 < x < 1$ a binomial random variable with parameters $k$ and $x$ is the sum of $k$ independent $\{0,1\}$-random variables $X_1,\ldots,X_k$ such that $\mathrm{Prob}\{X_i = 1\} = x$ for $i = 1,\ldots,k$. Thus the number of edges in a graph in $\mathcal{G}_n$ is a binomial random variable with parameters $\binom{n}{2}$ and $p$.

We consider also the set $\mathcal{G}_n^*$ of all graphs with vertices the sets of a partition of $\{1,\ldots,n\}$. If $k$ is an integer we shall often confuse $k$ and $\{k\}$. Thus for example we may say that $\mathcal{G}_n \subseteq \mathcal{G}_n^*$. The use of sets to label vertices is simply a notational convenience.

We shall sometimes make statements involving such phrases as 'for almost all graphs in $\mathcal{G}_n$'. For example Lemma 5.2 below states that for almost all graphs $G_n$ in $\mathcal{G}_n$

$$\chi(G_n) \geq 1/2\, n / \log n \quad .$$

This simply means that

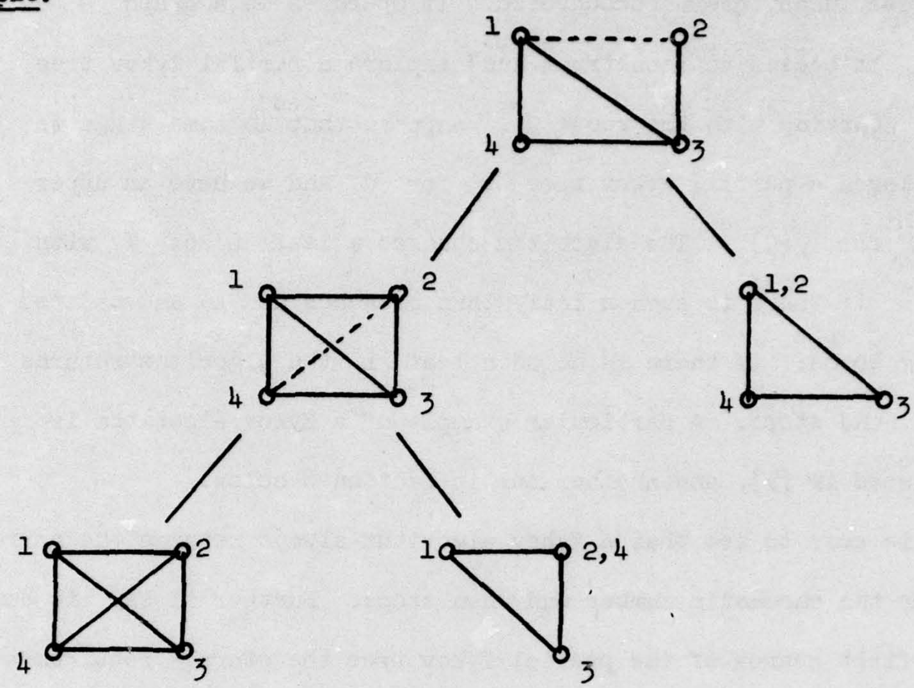
$$\mathrm{Prob}\{G \in \mathcal{G}_n : \chi(G_n) \geq 1/2\, n / \log n\} \to 1 \quad \text{as } n \to \infty .$$

We now move on towards our definitions of Zykov trees and Zykov algorithms. Suppose that $x$ and $y$ are non-adjacent vertices in a graph $H$ in $\mathscr{G}_n^*$. Following [5] we define the reduced graphs $H'_{xy}$ and $H''_{xy}$. The former $H'_{xy}$ is obtained from $H$ by simply adding an edge joining $x$ and $y$; and the latter $H''_{xy}$ is obtained from $H$ by replacing the vertices $x$ and $y$ by a single new vertex $x \cup y$ adjacent to each vertex to which $x$ or $y$ was adjacent. We say that $H'_{xy}$ and $H''_{xy}$ are obtained from $H$ by an 'edge-addition' and a 'vertex-contraction' respectively. In any proper coloring of $H$ either $x$ and $y$ have different colors or they have the same color. Thus we have the well known result (see [15]) that

$$\chi(H) = \min\{\chi(H'_{xy}), \chi(H''_{xy})\} \quad . \tag{2.1}$$

Suppose that we have a graph $H$ in $\mathscr{G}_n^*$ which is itself a leaf in a binary tree. Then branching at $H$ involves choosing non-adjacent vertices $x$ and $y$ in $H$ and giving $H$ the leftson $H'_{xy}$ and the rightson $H''_{xy}$. Of course we cannot branch at $H$ if $H$ is complete. Now let $G$ be a graph in $\mathscr{G}_n$. If we start with the single node $G$, the root of our binary tree, and branch repeatedly we obtain a partial Zykov tree for $G$. By (2.1) we know that $\chi(G)$ is the minimum value of $\chi(L)$ over all leaves $L$ of any partial Zykov tree for $G$. A Zykov tree for $G$ is a partial Zykov tree in which each leaf is a complete graph. We give below an example of a Zykov tree for a graph in $\mathscr{G}_4$. (See also [2] Chapter 15, [5].)

Example.




We have now described the 'branching' process to be used in our branch-and-bound algorithms. The 'bounding' process depends on the obvious result that for any graph G

$$\chi(G) \geq \omega(G) \quad . \tag{2.2}$$

A <u>Zykov algorithm</u> is a branch-and-bound algorithm for determining the chromatic number of a graph, using branch and bound processes as described above. Such an algorithm has a subroutine for determining for each graph H a lower bound $\omega'(H)$ for $\omega(H)$ (for example by finding a complete subgraph of H ). Also it maintains a current best upper bound for the chromatic number, which is always at most the number

of vertices in any graph encountered. It operates on a graph $G$ as follows. It begins to (construct and) explore a partial Zykov tree for $G$, starting with the root $G$. Suppose that at some stage we have explored a partial Zykov tree $T$ for $G$ and we have an upper bound $b$ for $\chi(G)$. The algorithm chooses a leaf $L$ of $T$ with $\omega'(L) < b$ if there is such a leaf, then branches at $L$ and updates the upper bound: if there is no such leaf $L$ the algorithm returns $\chi(G) = b$ and stops. A particular example of a Zykov algorithm is investigated in [5], and another one in Section 6 below.

It is easy to see that a Zykov algorithm always returns the correct value for the chromatic number and then stops. Further if say it conducts a depth-first search of the partial Zykov tree the storage requirement need only be say $O(n^3)$. The problem is that Zykov algorithms are very slow, even if we suppose that the subroutine can always determine $\omega(H)$ exactly and without cost, and that we can always start with the upper bound at the actual value of the chromatic number. (Both these suppositions are of course rather unlikely, since we would be solving NP-complete problems [1].)

Given a Zykov tree $Z$ for a graph $G$ the corresponding pruned Zykov tree consists simply of the root $G$ if $\omega(G) = \chi(G)$ and otherwise is the unique maximal rooted subtree of $Z$ containing as internal nodes precisely the nodes $H$ of $Z$ with $\omega(H) < \chi(G)$. Any Zykov algorithm must explore at least some proved Zykov tree for $G$. We shall prove that pruned Zykov trees are usually very large and thus that Zykov algorithms are usually very slow.

Finally let us establish some notation. We let $\mathbb{N}$ denote the set of positive integers and $\mathbb{Z}$ the set of non-negative integers. For any real number $x$ we let $\lceil x \rceil$ denote the least integer not less than $x$ and $\lfloor x \rfloor$ denote the greatest integer not more than $x$. Recall that $q$ is a constant with $0 < q < 1$ (except that in part of Section 3 we allow $q$ to vary). All logarithms are to the base $1/q$ unless otherwise indicated.

## 3. Preliminary Results.

In this section we present some necessary preliminary lemmas, which may be of interest in their own right. Lemma 3.1 is well known and is used only in the proof of Lemma 3.2, which is the most used result in this section. The remaining results, Lemmas 3.3 to 3.6 concern the 'bounded sequential coloring algorithm', and are needed here only for the 'converse' results in Sections 4 and 5 and for Section 8.

Let $m, n \in \mathbb{N}$ and let $Q = (S_1, \ldots, S_m)$ be a family of pairwise disjoint subsets of $\{1, \ldots, n\}$. We say that $Q$ is proper for a graph $G$ in $\mathcal{G}_n$ if no two adjacent vertices of $G$ are in the same set $S_i$ in $Q$. For each graph $G$ in $\mathcal{G}_n$ we define a 'contracted' graph $G_Q$ as follows: the graph $G_Q$ has vertices $S_1, \ldots, S_m$ and an edge between the vertices $S_i$ and $S_j$ if and only if there is an edge in $G$ between some vertex in the set $S_i$ and some vertex in the set $S_j$. Clearly $G_Q$ may be formed from $G$ by a sequence of vertex-contractions if and only if $Q$ is proper for $G$.

Now let $m, n \in \mathbb{N}$ and let $Q$ be a partition of $\{1, \ldots, n\}$ into $m$ sets. It seems reasonable to think that we are likely to have more edges in $G_Q$ the more equal in size are the sets in $Q$. We prove below that this is true.

For any random variable $X$ we let $F_X$ denote its distribution function, that is

$$F_X(t) = \text{Prob}\{X \leq t\}$$

for each real number $t$. Given two random variables $X$ and $Y$ we write $X \leq Y$ in distribution if $F_X(t) \geq F_Y(t)$ for each real number $t$.

Lemma 3.1. Suppose that $X, Y, Z$ are random variables, that $X \leq Y$ in distribution, and that both the pairs $X, Z$ and $Y, Z$ are independent. Then $X+Z \leq Y+Z$ in distribution.

Proof. For any real number $t$,

$$\begin{aligned} F_{X+Z}(t) &= \int F_X(t-u)\, dF_Z(u) \\ &\geq \int F_Y(t-u)\, dF_Z(u) = F_{Y+Z}(t) \quad . \quad \square \end{aligned}$$

Let $m, n \in \mathbb{N}$ and suppose that $m$ is fixed. For each real number $q$ with $0 < q < 1$, let $N(q)$ be a binomial random variable with parameters $\binom{m}{2}$ and $(1-q)$, and for each partition $Q$ of $\{1,\ldots,n\}$ let $N(n,Q,q)$ be the number of edges in the contracted graph $G_Q$ for graphs $G$ in $\mathcal{G}_n$ with edge-probability $(1-q)$.

Lemma 3.2. For each partition $Q$ of $\{1,\ldots,n\}$ into $m$ sets we have

$$N(n,Q,q) \leq N(q^{(n/m)^2}) \quad \text{in distribution.} \tag{3.1}$$

Proof. We may of course assume that $m \geq 2$. We shall prove first that for each partition $Q$ of $\{1,\ldots,n\}$ into $m$ sets we have

$$N(n,Q,q) \leq N(q^{\lceil n/m \rceil^2}) \quad \text{in distribution.} \tag{3.2}$$

Let $Q = (S_1,\ldots,S_m)$ be a partition of $\{1,\ldots,n\}$ into $m$ sets; let $s_i = |S_i|$ for $i = 1,\ldots,m$; and suppose that $s_1+1 \leq s_2-1$. Let $v \in S_2$ and let $Q'$ be the partition obtained from $Q$ by switching $v$ from $S_2$ to $S_1$. In this part of the proof of the lemma both $n$ and $q$ will be fixed. Denote $N(n,Q,q)$ and $N(n,Q',q)$ by $N_Q$ and $N_{Q'}$ respectively. In order to prove (3.2) it is sufficient to prove that

$$N_Q \leq N_{Q'} \quad \text{in distribution.} \tag{3.3}$$

Consider first the case $m = 2$, when $N_Q$ and $N_{Q'}$ may take only the values 0 and 1. Clearly

$$\mathrm{Prob}\{N_Q = 1\} = 1 - q^{s_1 s_2} \leq 1 - q^{(s_1+1)(s_2-1)} = \mathrm{Prob}\{N_{Q'} = 1\} ,$$

and (3.3) follows.

Suppose now that $m \geq 3$. Let $R$ and $R'$ be the partitions $Q$ and $Q'$ respectively with the last set deleted. By induction we may assume that $N_R \leq N_{R'}$ in distribution (in an obvious notation). Let $D$ and $D'$ be random variables giving the degree of the 'last' vertex in $G_Q$ and $G_{Q'}$ respectively. Then $N_R$ and $N_{R'}$ are independent of $D$ and $D'$, and $N_Q = N_R+D$ and $N_{Q'} = N_{R'}+D'$. Hence by Lemma 3.1 in order to prove (3.3) and so (3.2) it is sufficient to prove that

$$D \leq D' \quad \text{in distribution.} \tag{3.4}$$

For $i = 1,\ldots,m-1$ let $X_i = 1$ if $S_i$ and $S_m$ are adjacent as vertices in $G_Q$ and let $X_i = 0$ otherwise. Define random variables $X_i'$ from $Q'$ in a similar manner. Then the random variables $X_1,\ldots,X_{m-1}$ are independent and sum to $D$; the random variables $X_1',\ldots,X_{m-1}'$ are independent and sum to $D'$; and $X_i = X_i'$ for $i = 3,\ldots,m-1$. Hence by Lemma 3.1 in order to prove (3.4) (and so (3.3) and so (3.2)) it is sufficient to prove that

$$X_1 + X_2 \leq X_1' + X_2' \quad \text{in distribution.} \tag{3.5}$$

Note first that $X_1+X_2$ and $X_1' + X_2'$ may take only the values $0, 1, 2$. Now

$$\mathrm{Prob}\{X_1+X_2 \geq 1\} = 1 - q^{(s_1+s_2)s_m} = \mathrm{Prob}\{X_1' + X_2' \geq 1\} .$$

Also

$$\begin{aligned}\mathrm{Prob}\{X_1+X_2 \ge 2\} &= \mathrm{Prob}\{X_1 = 1\}\mathrm{Prob}\{X_2 = 1\}\\ &= (1-q^{s_1 s_m})(1-q^{s_2 s_m})\\ &= 1-q^{s_1 s_m}-q^{s_2 s_m}+q^{(s_1+s_2)s_m}\ ,\end{aligned}$$

and similarly

$$\begin{aligned}\mathrm{Prob}\{X_1'+X_2' \ge 2\} &= \mathrm{Prob}\{X_1' = 1\}\mathrm{Prob}\{X_2' = 1\}\\ &= (1-q^{(s_1+1)s_m})(1-q^{(s_2-1)s_m})\\ &= 1-q^{(s_1+1)s_m}-q^{(s_2-1)s_m}+q^{(s_1+s_2)s_m}\ .\end{aligned}$$

Now let $t = q^{s_m}$, so that $0 \le t \le 1$. Then

$$\begin{aligned}&\mathrm{Prob}\{X_1'+X_2' \ge 2\} - \mathrm{Prob}\{X_1+X_2 \ge 2\}\\ &\quad = t^{s_1}+t^{s_2}-t^{(s_1+1)}-t^{(s_2-1)}\\ &\quad = (1-t)(t^{s_1}-t^{s_2-1})\ ,\end{aligned}$$

and this last expression is non-negative, since $s_1 \le s_2-1$. But this completes the proof of (3.5) and so of (3.2). We now use (3.2) to prove (3.1).

Given a set $S$ of positive integers and a positive integer $k$ let $kS$ be the set of positive integers $i$ such that $\lceil i/k \rceil$ is in $S$. Given a partition $Q = (S_1, S_2, \ldots, S_m)$ of $\{1,2,\ldots,n\}$ for some integer $n$ let $kQ$ be the partition $(kS_1, kS_2, \ldots, kS_m)$ of $\{1,2,\ldots,kn\}$. For example if $Q$ is the partition $(\{1,2\},\{3\})$ of $\{1,2,3\}$ then $2Q$ is the partition $(\{1,2,3,4\},\{5,6\})$ of $\{1,2,\ldots,6\}$.

Let $n \in \mathbb{N}$, let $Q$ be a partition of $\{1,2,\ldots,n\}$ into $m$ sets and let $q$ be a real number with $0 < q < 1$. Let $\mathcal{G}_{n,1-q}$ denote the set $\mathcal{G}_n$ with edge-probabilities $1-q$. Then $N(n,Q,q)$ is the sum of $\binom{m}{2}$ independent $\{0,1\}$ random variables $X_{ij}$ $(1 \le i < j \le m)$ such that

$$\begin{aligned}\text{Prob}\{X_{ij}=1\} &= \text{Prob}\{G \in \mathcal{G}_{n,1-q}: \text{some vertex in } S_i \text{ is adjacent to some vertex in } S_j\} \\ &= 1-q^{|S_i||S_j|}.\end{aligned}$$

Let $k$ be a positive integer. Then $N(kn, kQ, q^{1/k^2})$ is the sum of $\binom{m}{2}$ independent $\{0,1\}$ random variables $Y_{ij}$ $(1 \le i < j \le m)$ such that

$$\begin{aligned}\text{Prob}\{Y_{ij}=1\} &= \text{Prob}\{G \in \mathcal{G}_{kn,1-q^{1/k^2}}: \text{some vertex in } kS_i \text{ is adjacent to some vertex in } kS_j\} \\ &= 1-(q^{1/k^2})^{|kS_i||kS_j|} \\ &= 1-q^{|S_i||S_j|}.\end{aligned}$$

Hence for each positive integer $k$,

$$N(n,Q,q) = N(kn, kQ, q^{1/k^2}) \quad \text{in distribution.} \tag{3.6}$$

By (3.2) and (3.6) for each $k \in \mathbb{N}$ we have that in distribution

$$\begin{aligned}N(n,Q,q) &= N(kn, kQ, q^{1/k^2}) \\ &\le N\left(q^{\frac{1}{k^2}\left\lceil\frac{kn}{m}\right\rceil^2}\right).\end{aligned}$$

But $\frac{1}{k^2}\left\lceil\frac{kn}{m}\right\rceil^2 \to \left(\frac{n}{m}\right)^2$ as $k \to \infty$, and so clearly (3.1) holds. This completes the proof of Lemma 3.2. $\square$

We define an algorithm related to the sequential algorithm (SA) for coloring graphs (see [8], [9], [13]) and which we call the <u>bounded sequential algorithm</u> (BSA). We shall look at graphs $G$ in $\mathcal{G}_n$ for some $n$ in $\mathbb{N}$. Suppose that we have a positive integer $s$. The BSA (bounded at $s$) acts on each graph $G$ in the same way as the SA, except that we allow each color set to contain at most $s$ elements. Thus the BSA (bounded at $s$) colors vertex 1 with color 1 and then colors the remaining vertices in increasing order, coloring vertex $i$ with color $j$ if $j$ is the least positive integer such that vertex $i$ is not adjacent to any vertex already colored $j$ and such that there are at most $(s-1)$ vertices already colored $j$.

Suppose now that we have also a positive integer $t$. For each graph $G$ in $\mathcal{G}_n$ we shall be interested in the family $Q(G)$ $(= Q_{s,t}(G))$ consisting of the first $t$ color sets constructed by the BSA (bounded at $s$); and more interested in the contracted graph $G' = G_{Q(G)}$. We say that a family $Q(G)$ as above is <u>full</u> if each of the $t$ sets contains the full $s$ elements.

<u>Lemma 3.3</u>. Let $N$ be a binomial random variable with parameters $\binom{t}{2}$ and $q^{s^2}$. Then for each non-negative integer $k$

$$\mathrm{Prob}\{G_n' \text{ misses at most } k \text{ edges}\} \geq \mathrm{Prob}\{N \leq k\}\mathrm{Prob}\{Q(G_n) \text{ is full}\}.$$

<u>Proof</u>. Let $\mathcal{R}$ $(= \mathcal{R}(n,s,t))$ be the collection of all the families $Q(G)$ for graphs $G$ in $\mathcal{G}_n$. Thus $\mathcal{R}$ is the collection of all families $(S_1,\ldots,S_t)$ of $t$ disjoint subsets of $\{1,\ldots,n\}$ each of size at most $s$ and such that for each index $i$ in $\{1,\ldots,t\}$ and each vertex $v$ in a set with index greater than $i$, if $|S_i| < s$ or $v < u$ for some vertex $u$ in $S_i$ then $v > u'$ for some vertex $u'$ in $S_i$.

Let $Q = (S_1, \ldots, S_t)$ be a family in $\mathcal{R}$. Let $X$ be the set of graphs $G$ in $\mathcal{G}_n$ such that no two vertices adjacent in $G$ lie in the same set $S_i$, and let $Y$ be the set of graphs $G$ in $\mathcal{G}_n$ such that for each index $i$ in $\{1, \ldots, t\}$ and each vertex $v$ in a set with index greater than $i$, if $|S_i| < s$ or $v < u$ for some vertex $u$ in $S_i$ then $v$ is adjacent in $G$ to some vertex $u'$ in $S_i$ with $v > u'$. Then

$$\{G \in \mathcal{G}_n : Q(G) = Q\} = X \cap Y .$$

Now clearly in distribution we have

$$|E(G_Q)| \leq |E(G_Q)| \quad \text{given } G \in Y ,$$

and conditioning on $X$ does not affect the distribution of the number of edges in $G_Q$. Thus in distribution

$$|E(G_Q)| \leq |E(G_Q)| \quad \text{given } Q(G) = Q . \tag{3.7}$$

But now for each $k \in \mathbb{Z}$,

$$\begin{aligned}
&\mathrm{Prob}\{G' \text{ misses at most } k \text{ edges}\} \\
&= \mathrm{Prob}\{|E(G')| \geq \binom{t}{2} - k\} \\
&= \sum_{Q \in \mathcal{R}} \mathrm{Prob}\{|E(G_Q)| \geq \binom{t}{2} - k \mid Q(G) = Q\}\mathrm{Prob}\{Q(G) = Q\} \\
&\geq \sum_{Q \in \mathcal{R}} \mathrm{Prob}\{|E(G_Q)| \geq \binom{t}{2} - k\}\mathrm{Prob}\{Q(G) = Q\} \quad \text{(by (3.7))} \\
&\geq \sum_{\substack{Q \in \mathcal{R} \\ Q \text{ full}}} \mathrm{Prob}\{\binom{t}{2} - |E(G_Q)| \leq k\}\mathrm{Prob}\{Q(G) = Q\} \\
&= \mathrm{Prob}\{N \leq k\}\mathrm{Prob}\{Q(G) \text{ full}\} . \quad \square
\end{aligned}$$

Lemma 3.4. For any positive integers $n, s, t$ with $st \le n$

$$\mathrm{Prob}\{Q(G_n) \text{ not full}\} \le n(1-q^{s-1})^{n/s-t+1} .$$

Proof. For each graph $G$ in $\mathcal{G}_n$ and for $i = 1,\dots,t$ let $S_i(G)$ denote the i-th set in $Q(G)$. Then

$$\{Q(G_n) \text{ not full}\} = \bigcup_{i=1}^{t} \{|S_i(G_n)| < s\} .$$

Now for each $k \le n$ in $\mathbb{N}$ and each graph $G_n$ in $\mathcal{G}_n$ let $\sigma_k(G_n)$ denote the number of vertices of $G_n$ amongst the first $k$ which the SA colors with the first color (see [8]). Then

$$\begin{aligned}
\mathrm{Prob}\{Q(G_n) \text{ not full}\} &\le \sum_{i=1}^{t} \mathrm{Prob}\{|S_i(G_n)| < s\} \\
&\le \sum_{i=1}^{t} \mathrm{Prob}\{\sigma_{n-(i-1)s}(G_n) < s\} \\
&\le \sum_{i=1}^{t} s(1-q^{s-1})^{n/s-(i-1)} \qquad \text{(see [8])} \\
&\le st(1-q^{s-1})^{n/s-(t-1)} \\
&\le n(1-q^{s-1})^{n/s-t+1} . \qquad \square
\end{aligned}$$

Lemma 3.5. Let $\varepsilon > 0$ and let $s$ and $t$ be functions from $\mathbb{N}$ to $\mathbb{N}$ such that $s(n) \le (1-\varepsilon)\log n$ and $s(n)t(n) \le (1-\varepsilon)n$ for each $n$ in $\mathbb{N}$. Then

$$\mathrm{Prob}\{Q(G_n) \text{ full}\} \to 1 \quad \text{as } n \to \infty . \tag{3.8}$$

If further $s(n) \ge (2 \log n)^{1/2}$ for each $n$ in $\mathbb{N}$ then

$$\mathrm{Prob}\{G_n' \text{ complete}\} \to 1 \quad \text{as } n \to \infty . \tag{3.9}$$

**Proof.** By Lemma 3.4

$$\log \text{Prob}\{Q(G_n) \text{ not full}\} \leq \log n - \varepsilon n/\log n \cdot \log e \quad n^{-(1-\varepsilon)}$$

$$\to -\infty \quad \text{as} \quad n \to \infty ,$$

and so (3.8) holds. Now suppose that $s(n) \geq (2 \log n)^{1/2}$ for each $n$ in $\mathbb{N}$. If $N$ is as defined in Lemma 3.3 then

$$\begin{aligned} 0 &> \log \text{Prob}\{N = 0\} \\ &= \binom{t}{2} \log(1 - q^{s^2}) \\ &\geq \frac{n^2}{2 \log n} \log(1 - 1/n^2) \\ &\to 0 \quad \text{as} \quad n \to \infty . \end{aligned}$$

Hence

$$\text{Prob}\{N = 0\} \to 1 \quad \text{as} \quad n \to \infty . \tag{3.10}$$

Now (3.9) follows from (3.8), (3.10) and Lemma 3.3. □

**Lemma 3.6.** Let $\varepsilon > 0$. Then for almost all graphs $G$ in $\mathcal{G}_n$ there is a proper partition $R$ of $G$ into at least $(1-\varepsilon)n(2 \log n)^{-1/2}$ sets such that the contracted graph $G_R$ is complete.

**Proof.** Let $s(n) = \lceil (2 \log n)^{1/2} \rceil$ and $t(n) = \lceil (1-\varepsilon)n(2 \log n)^{-1/2} \rceil$ for each $n$ in $\mathbb{N}$. Then by Lemma 3.5

$$\text{Prob}\{G_n' \text{ complete}\} \to 1 \quad \text{as} \quad n \to \infty . \tag{3.11}$$

Now with each graph $G$ in $\mathcal{G}_n$ we shall associate a proper partition $R(G)$ related to the proper family $Q(G)$, and the contracted graph $G^* = G_{R(G)}$ related to the contracted graph $G' = G_{Q(G)}$. Consider a graph $G$ in $\mathcal{G}_n$. Suppose that the vertices of $G$ not in any set in $Q(G)$ are $v_1, \ldots, v_j$ in increasing order. For $i = 1, \ldots, j$ in turn

add the vertex $v_i$ to the first possible set in $Q(G)$ (that is, to the first set in $Q(G)$ such that $v_i$ is not adjacent to any vertex in the set) and if we cannot add $v_i$ to any already present set in $Q(G)$ then we add to $Q(G)$ a new singleton set $\{v_i\}$. In this way we construct a proper partition $R(G)$ of $G$ with at least $t$ sets. Let $G^*$ be the contracted graph $G_{R(G)}$. Then clearly the number of edges missing in $G^*$ is at most the number of edges missing in $G'$. Hence in particular we have by (3.11) that

$$\text{Prob}\{G_n^* \text{ complete}\} \to 1 \quad \text{as} \quad n \to \infty . \quad \square$$

Lemmas 3.5 and 3.6 are in convenient forms for the present purposes: they clearly are not in their strongest forms.

## 4. Zykov Trees.

In this section we investigate the sizes of Zykov trees. We have three main reasons for doing this. Firstly the sizes of Zykov trees are of interest in their own right, for example if we wish to determine the chromatic polynomial of a graph ([2], Chapter 15); secondly some knowledge of the sizes of Zykov trees helps us to interpret results on the sizes of pruned Zykov trees; and thirdly some of the arguments which we use here are similar to those we use for proved Zykov trees in the next section.

There are two theorems in this section. The first shows in particular that every Zykov tree for a given graph has the same size, that is the same number of nodes. Given a graph $G$ let us denote by $C(G)$ the number of proper partitions of $G$ (that is, the number of colorings of $G$ with 'color indifference').

**Theorem 4.1.** Every Zykov tree $T$ for a graph $G$ has $2C(G)-1$ nodes.

**Proof.** It is not hard to check that the vertex sets of the leaves of $T$ are in 1-1 correspondence with the proper partitions of $G$. □

The next theorem gives asymptotic results which by Theorem 4.1 above may be stated in terms either of the size of Zykov trees for a graph $G$ or of the number $C(G)$ of proper partitions of $G$. We choose to state them in terms of the latter. It is convenient to separate out part of the proof as a lemma.

For every $n$ in $\mathbb{N}$ and $\ell, r$ in $\mathbb{Z}$ let $T_n(\ell,r)$ be the set of graphs $G$ in $\mathcal{G}_n$ such that in every Zykov tree for $G$ if we start at the root $G$ we can always make $\ell(n)$ left turns and $r(n)$ right turns without reaching a leaf. If a graph $G$ is in $T_n(\ell,r)$

then certainly every Zykov tree for $G$ has at least $\binom{\ell+r}{r}$ nodes. We wish to choose the functions $\ell$ and $r$ so that Prob $T_n(\ell,r) \to 1$ as $n \to \infty$ and $\binom{\ell+r}{r}$ is as large as possible.

Lemma 4.2. There exist functions $\ell$ and $r$ from $\mathbb{N}$ to $\mathbb{N}$ such that

$$\text{Prob } T_n(\ell,r) \to 1 \quad \text{as } n \to \infty \tag{4.1}$$

and

$$\log\binom{\ell+r}{r} \geq n(\log n - 3(\tfrac{1}{2}\log n)^{2/3}) \quad \text{for } n \text{ sufficiently large} \tag{4.2}$$

For example we may take $\ell$ and $r$ so that

$$\ell(n) = n^{2-(2^{-2/3}+o(1))(\log n)^{-1/3}} \tag{4.3}$$

and

$$r(n) = \lfloor n(1-(\tfrac{1}{2}\log n)^{-1/3})\rfloor \ . \tag{4.4}$$

Proof. Let $\ell$ and $r$ be functions from $\mathbb{N}$ to $\mathbb{N}$, such that $\ell(n) \leq \binom{n}{2}$ and $r(n) \leq n-1$, which we shall choose below. For each $n$ in $\mathbb{N}$ let $m(n) = n-r(n)$, let $x(n) = n/m$ and let $k(n) = \binom{m}{2}$. We shall choose $r$ so that $x(n) \to \infty$ as $n \to \infty$ but $x(n) = o((\log n)^{1/2})$. Let $\mathcal{R}_n$ denote the set of partitions of $\{1,2,\ldots,n\}$ into at least $m$ non-empty sets. Then the complement $\bar{T}_n(\ell,r)$ of $T_n(\ell,r)$ in $\mathcal{G}_n$ satisfies

$$\bar{T}_n(\ell,r) = \bigcup_{Q\in\mathcal{R}_n} \{G\in\mathcal{G}_n : Q \text{ proper for } G \text{ and } G_Q \text{ misses at most } \ell \text{ edges}\}$$

$$\subseteq \bigcup_{Q\in\mathcal{R}_n} \{G\in\mathcal{G}_n : G_Q \text{ misses at most } \ell \text{ edges}\} \ . \tag{4.5}$$

Let $N$ be a binomial random variable with parameters $k$ and $q^{x^2}$. Then by Lemma 3.2 for each partition $Q$ in $\mathcal{R}_n$ we have

$$\text{Prob}\{G \in \mathcal{G}_n : G_Q \text{ misses at most } \ell \text{ edges}\} \le \text{Prob}\{N \le \ell\} \quad . \tag{4.6}$$

Now clearly $\mathcal{R}_n$ contains at most $n^n$ partitions and so by (4.5) and (4.6)

$$\text{Prob } \bar{T}_n(\ell, r) \le n^n \text{ Prob}\{N \le \ell\} \quad . \tag{4.7}$$

We shall use (4.7) to ensure that $\text{Prob } \bar{T}_n(\ell, r) \to 0$ as $n \to \infty$, and so clearly we must take $\ell < E[N]$ (at least for large $n$). We let

$$\ell(n) \sim \frac{1}{2} E[N] = \frac{1}{2} k q^{x^2} \sim n^{2+o(1)} \quad . \tag{4.8}$$

Now

$$\begin{aligned} \text{Prob}\{N \le \ell\} &= \sum_{i=0}^{\ell} \binom{k}{i} (q^{x^2})^i (1 - q^{x^2})^{k-i} \\ &\le (\ell+1) \binom{k}{\ell} q^{x^2 \ell} (1 - q^{x^2})^{k-\ell} \end{aligned} \tag{4.9}$$

(for $n$ sufficiently large that $\ell(n) \le E[N]$ ).

Now by (4.7), (4.8) and (4.9)

$$\begin{aligned} &\log \text{Prob } \bar{T}_n(\ell, r) \\ &\le n \log n + \ell \log k - \ell \log \ell + \ell \log e - x^2 \ell - (k-\ell) \log e\, q^{x^2} + O(\log n) \\ &= n \log n + \ell(\log k - (-\log 2 + \log k - x^2) + \log e - x^2 - 2 \log e + o(1)) \\ &= n \log n - \ell(\log e - \log 2 + o(1)) \\ &\to -\infty \quad \text{as } n \to \infty \end{aligned}$$

Hence (4.1) holds. It remains to choose $r$. Now

$$\begin{aligned} \log \binom{\ell+r}{r} &\ge (n-m)(\log \ell - \log n) \\ &= n(1-x^{-1})(\log n - 2 \log x - x^2 + O(1)) \\ &\ge n(\log n - x^{-1} \log n - x^2 + x(1 + o(1))) \quad . \end{aligned} \tag{4.10}$$

Let $y(n) = (\frac{1}{2} \log n)^{1/3}$ and $m(n) = \lceil n/y \rceil$. Now $x = n/m$ and so

$$y \geq x \geq n(n/y + 1)^{-1} .$$

Thus

$$\log n/x + x^2 \leq (\log n)/y + (\log n)/n + y^2$$
$$= 3y^2 + o(1) .$$

Hence by (4.10)

$$\log\binom{\ell+r}{r} \geq n(\log n - 3(\frac{1}{2} \log n)^{2/3} + (2^{-1/3} + o(1))(\log n)^{1/3})$$
$$\geq n(\log n - 3(\frac{1}{2} \log n)^{2/3})$$

for $n$ sufficiently large. Thus we have proved (4.1). From the above we may easily check (4.3) and (4.4). This completes the proof of Lemma 4.2. □

<u>Theorem 4.3</u>. (1) For every graph $G_n$ in $\mathcal{G}_n$

$$\log C(G_n) \leq \log C(\emptyset_n) = n(\log n - \log\log n - \log e + o(1)) ,$$

where $\emptyset_n$ is the graph on $n$ vertices with no edges.

(2) The expected value $E[C_n]$ of $C(G_n)$ for graphs $G_n$ in $\mathcal{G}_n$ satisfies

$$\log E[C_n] = n(\log n - (2 \log n)^{1/2} - \frac{1}{2} \log\log n + O(1)) .$$

(3) For almost all graphs $G_n$ in $\mathcal{G}_n$

$$n(\log n - 3(\frac{1}{2} \log n)^{2/3}) \leq \log C(G_n) \leq n(\log n - (2 \log n)^{1/2}) .$$

<u>Proof</u>. (1) The first part follows easily from the observation that $C(\emptyset_n)$ is simply the number of partitions of $\{1,\ldots,n\}$ .

(2) We first show that

$$\log E[C_n] \geq n(\log n - (2 \log n)^{1/2} - \frac{1}{2} \log \log n + O(1)) \quad . \tag{4.11}$$

Let $d$ be a function from $\mathbb{N}$ to $\mathbb{N}$ such that $d(n) \to \infty$ as $n \to \infty$ but say $d(n) = O(n/\log n)$. We shall choose $d$ below. Let $\mathcal{R}_n$ be the set of partitions of $\{1,\dots,n\}$ into $k = \lfloor n/d \rfloor$ sets each of size $d$ and (possibly) the $(n-kd)$ singleton set $\{kd+1\},\dots,\{n\}$. Then the number of partitions in $\mathcal{R}_n$ equals

$$\frac{(kd)!}{k!(d!)^k} \geq \frac{(n-d)!}{(n/d)!(d!)^{n/d}}$$

and the probability that a partition in $\mathcal{R}_n$ is proper equals

$$q^{\binom{d}{2}k} \geq q^{\frac{1}{2}nd} \quad .$$

Hence the logarithm of the expected number of proper partitions in $\mathcal{R}_n$ is at least

$$(n-d) \log(n-d) - (n/d) \log(n/d) - (n/d)(d \log d) - \frac{1}{2} nd + O(n)$$

$$= n(\log n - \log n/d - \log d - \frac{1}{2} d + O(1)) \quad . \tag{4.12}$$

Now let

$$f_n(x) = \log n/x + \log x + \frac{1}{2} x$$

for $x > 0$. Then $f_n(x)$ achieves a unique minimum for $x > 0$ at $x = (2 \log n + 1)^{1/2} - 1$ and this minimum equals

$$(2 \log n)^{1/2} + \frac{1}{2} \log \log n + O(1) \quad . \tag{4.13}$$

We set $d(n) = \lfloor(2 \log n)^{1/2}\rfloor$ for $n \in \mathbb{N}$ and find that the right hand side in (4.12) equals

$$n(\log n - (2 \log n)^{1/2} - \frac{1}{2} \log \log n + O(1)) \quad .$$

Hence certainly (4.11) holds.

We now show that

$$\log E[C_n] \leq n(\log n - (2 \log n)^{1/2} - \frac{1}{2} \log \log n + O(1)) \quad . \tag{4.14}$$

The inequalities (4.11) and (4.14) of course prove the second part of the theorem.

Let $k = k(n)$ be an integer $i$ such that the expected number of proper partitions into $i$ non-empty sets is a maximum. Then clearly $E[C_n]$ is at most $n$ times the expected number of proper partitions into $k$ non-empty sets. Let $d = d(n) = n/k$ . (Thus $d(n)$ is not necessarily an integer.)

Let $Q = (S_1, \ldots, S_k)$ be a partition of $\{1, \ldots, n\}$ and let $s_i = |S_i|$ for $i = 1, \ldots, k$ . Then as in [8] we see that the probability that $Q$ is proper equals

$$\prod_{i=1}^{k} q^{\frac{1}{2} s_i(s_i - 1)} = q^{\frac{1}{2}(\sum s_i^2 - n)} \leq q^{\frac{1}{2}(n^2/k - n)} \quad .$$

Also the number of partitions of $\{1, \ldots, n\}$ into $k$ non-empty sets is at most $k^n/k!$ . Hence

$$E[C_n] \leq n \frac{k^n}{k!} q^{\frac{1}{2}(n^2/k - n)} \quad ,$$

and so

$$\log E[C_n] \leq n \log k - k \log k - \frac{1}{2}\frac{n^2}{k} + O(n)$$

$$= n \log n - n \log d - \frac{n}{d} \log n - \frac{1}{2} nd + O(n)$$

$$= n(\log n - f_n(d) + O(1)) \quad .$$

But by (4.13)

$$f_n(d) \geq (2 \log n)^{1/2} + \frac{1}{2} \log \log n + O(1)$$

and so we have proved (4.14).

(3) The left hand inequality in part (3) follows immediately from Lemma 4.2 and the discussion preceding it. Now clearly

$$\log E[C_n] \geq n(\log n - (2 \log n)^{1/2}) + \log \mathrm{Prob}\{\log C(G_n) \geq n(\log n - (2 \log n)^{1/2})\}$$

and so by part (2)

$$\log \mathrm{Prob}\{\log C(G_n) \geq n(\log n - (2 \log n)^{1/2})\}$$

$$\leq n(-\frac{1}{2} \log \log n + O(1))$$

$$\to -\infty \quad \text{as } n \to \infty \quad .$$

This proves the right hand inequality in part (3), and thus completes the proof of the theorem. □

There is a fairly large difference between the left and right hand sides in the third part of Theorem 4.3 above. The second part suggests that the right hand inequality in the third part may be quite good. It thus seems quite possible that the left hand inequality is rather weak. Recall that the left hand inequality follows from Lemma 4.2. Proposition 4.4 below shows that Lemma 4.2 is in a sense best possible. Proposition 4.4 corresponds to Proposition 5.7 in the next section. We do not prove Proposition 4.4 here: it may be proved along the lines of the proof of Proposition 5.7, using the results in Section 3.

<u>Proposition 4.4.</u> Let $\ell$ and $r$ be functions from $\mathbb{N}$ to $\mathbb{N}$ such that

$$\log\binom{\ell+r}{r} > n\{\log n - (3+o(1))(\tfrac{1}{2}\log n)^{2/3}\} \quad . \tag{4.15}$$

Then

$$\text{Prob } T_n(\ell,r) \to 0 \quad \text{as } n \to \infty \ .$$

Note that (4.15) above means that for any function $f$ such that

$$f(n) = n\{\log n - (3+o(1))(\tfrac{1}{2}\log n)^{2/3}\}$$

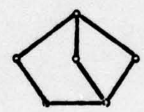
we have

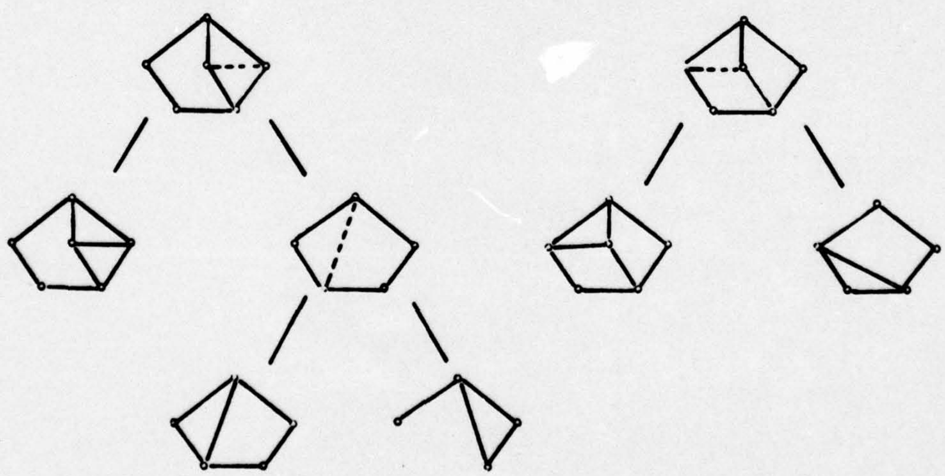
$$\log\binom{\ell+r}{r} > f(n) \quad \text{for } n \text{ sufficiently large.}$$

## 5. Pruned Zykov Trees.

In this section we investigate the size of pruned Zykov trees. We do not manage to find out as much about pruned Zykov trees as we found out about (unpruned) Zykov trees in the last section, but we are able to prove a greater than exponential lower bound. This result shows that Zykov algorithms for determining the chromatic number of a graph usually require more than exponential time.

We have seen that every Zykov tree for a given graph has the same size. Thus certainly if we have to construct a Zykov tree there is no point in spending time choosing a 'best' way of branching. The situation is quite different when we look at pruned Zykov trees. Two pruned Zykov trees for a given graph may have different sizes.

Example. Two pruned Zykov trees for  .



For every graph $G$ let $r(G)$ be the ratio of the greatest size to the smallest size for pruned Zykov trees for $G$ ; and for each $n$ in $\mathbb{N}$ let $r(n)$ be the maximum value of $r(G)$ over all graphs $G$ on $n$ vertices. Thus $r(n)$ is a measure of the possible variation in sizes of pruned Zykov trees for graphs on $n$ vertices.

For each graph $G$ on at most four vertices we have $\chi(G) = \omega(G)$ and so every pruned Zykov tree for $G$ has exactly one node. Thus

$$r(1) = r(2) = r(3) = r(4) = 1 .$$

The example above shows that $r(5) > 1$ , and by adding isolated vertices to a graph it is easy to see that $r(n)$ (strictly) increases from $n = 5$ onwards. Thus

$$r(n) > 1 \quad \text{for } n > 4 .$$

In fact $r(n)$ grows dramatically with $n$ .

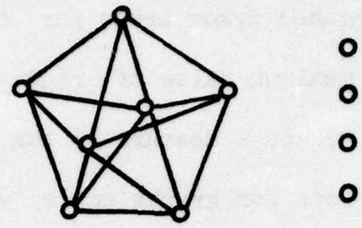
<u>Proposition 5.1.</u> $r(n) \geq n^{\frac{n}{2}(1+o(1))}$ .

We prove Proposition 5.1 by constructing for each integer $n \geq 7$ a graph $H_n'$ on $n$ vertices such that

$$r(H_n') \geq 2C(\emptyset_{\lfloor \frac{n}{2} \rfloor - 1}) - 1 = n^{\frac{n}{2}(1+o(1))} . \tag{5.1}$$

Here $C(\emptyset_k)$ is the number of partitions of a set of $k$ distinct elements (see Theorem 4.3).

First for each integer $k \geq 5$ let $H_k$ be the pentagon $C_5$ plus $(k-5)$ vertices adjacent to each other vertex. Thus $H_k$ is a 'wheel with $(k-5)$ axles': see the example below for $H_7$ . It is easy to check that $\omega(H_k) = k-3$ and $\chi(H_k) = k-2$ ; and that every pruned Zykov tree for $H_k$ has exactly three nodes. Now for each integer $n \geq 7$ let $H_n'$ be the graph $H_{\lceil \frac{n}{2} \rceil + 1}$ together with $\lfloor n/2 \rfloor - 1$ isolated vertices.



<u>Example</u>. $H'_{11}$ is $H_7$ plus 4 isolated vertices.

By branching within the large component of $H'_n$ we see that the smallest size of a pruned Zykov tree for $H'_n$ is 3. Now

$$\lfloor n/2 \rfloor - 1 \leq \lceil n/2 \rceil - 1 = \chi(H'_n) \quad .$$

Hence by branching first amongst the $\lfloor n/2 \rfloor - 1$ isolated vertices in $H'_n$ we see that the greatest size of a pruned Zykov tree for $H'_n$ is at least the size of an unpruned Zykov tree for the graph $\emptyset_{\lfloor n/2 \rfloor - 1}$ consisting of $\lfloor n/2 \rfloor - 1$ isolated vertices. But by *Theorem* 4.3 every Zykov tree for this graph has $2C(\emptyset_{\lfloor n/2 \rfloor - 1}) - 1$ nodes. We have now proved (5.1) and so completed the proof of Proposition 5.1. □

Note that if the isolated vertices are listed first then the marked Zykov algorithm will explore at least the large pruned Zykov tree for $H'_n$, and so the backtrack coloring algorithm will also do badly (see Section 6).

We now move on towards our main results. We need first a lemma concerning the chromatic number of a random graph, which is taken essentially from [8]. Recall that all logarithms are to the base $1/q$ unless otherwise indicated. A set of vertices in a graph G is <u>stable</u> if no two are adjacent, and the <u>stability number</u> $\alpha(G)$ is the greatest number of vertices in a stable set.

<u>Lemma 5.2</u>. For almost all graphs $G_n$ in $\mathcal{G}_n$

$$\chi(G_n) \geq \frac{1}{2} n / \log n \quad .$$

<u>Proof</u>. If $\chi(G_n) \leq \frac{1}{2} n/\log n$ then certainly the stability number $\alpha(G_n)$ of $G_n$ satisfies

$$\alpha(G_n) \geq n/\chi(G_n) \geq 2 \log n \quad .$$

But if we set $s(n) = \lceil 2 \log n \rceil$ then

$$\mathrm{Prob}\{\alpha(G_n) \geq s\} \leq \binom{n}{s} q^{\binom{s}{2}} \to 0 \quad \text{as} \quad n \to \infty \, .$$

Hence $\mathrm{Prob}\{\chi(G_n) \leq \frac{1}{2} n/\log n\} \to 0$ as $n \to \infty$. □

The following conjecture appears essentially in [8].

<u>Conjecture 5.3</u>. If $\varepsilon > 0$ then for almost all graphs $G_n$ in $\mathcal{G}_n$

$$\chi(G_n) \leq (\tfrac{1}{2} + \varepsilon) n / \log n \quad .$$

We need one more lemma in order to prove our main results. Suppose that we have a positive constant $\alpha$ and functions $\ell$ and $r$ from $\mathbb{N}$ to $\mathbb{Z}$. For each $n$ in $\mathbb{N}$ let $T_n^{\alpha}(\ell,r)$ be the set of graphs $G$ in $\mathcal{G}_n$ such that in every Zykov tree for $G$ whenever we start at the root $G$ and make $\ell(n)$ left turns and $r(n)$ right turns we do not encounter any node $H$ with $\omega(H) \geq \alpha\, \chi(G)$. (Compare with the definition of $T_n(\ell,r)$ preceding Lemma 4.2 in Section 4.) If $G$ is a graph in $T_n^{\alpha}(\ell,r)$ then certainly every Zykov tree for $G$ has at least $\binom{\ell+r}{r}$ nodes $H$ with $\omega(H) < \alpha\, \chi(G)$. Thus setting $\alpha = 1$ we see that if $G$ is in $T_n^{1}(\ell,r)$ then every pruned Zykov tree for $G$ has at least $\binom{\ell+r}{r}$ nodes. We wish to choose the functions $\ell$ and $r$ so that

Prob $T_n^\alpha(\ell,r) \to 1$ as $n \to \infty$ and $\binom{\ell+r}{r}$ is as large as possible.

Lemma 5.4. Let $\alpha$ be a positive constant. Then there exist functions $\ell$ and $r$ from $\mathbb{N}$ to $\mathbb{Z}$ such that

$$\text{Prob } T_n^\alpha(\ell,r) \to 1 \quad \text{as } n \to \infty \tag{5.2}$$

and

$$\log\binom{\ell+r}{r} \sim \alpha\, n\left(\frac{1}{27}\log n\right)^{1/2} . \tag{5.3}$$

For example we may take

$$\ell(n) = \lfloor n^{5/3}(\log n)^{-3} \rfloor \tag{5.4}$$

and

$$r(n) = \lfloor n(12 \log n)^{-1/2} \rfloor . \tag{5.5}$$

Lemma 5.4 above of course corresponds to Lemma 4.2 for (unpruned) Zykov trees, and we saw in Section 4 that Lemma 4.2 is in a sense best possible. At the end of this section we shall prove that Lemma 5.4 is also in a sense best possible.

Proof. Let $\ell$ and $r$ be functions from $\mathbb{N}$ to $\mathbb{N}$, which we shall choose later. Let $b(n) = \lfloor \frac{\alpha}{2} n(\log n)^{-1} \rfloor$, let $B_n$ be the set of graphs $G$ in $\mathcal{G}_n$ such that $\chi(G) \geq b(n)$, and let $B_n(\ell,r)$ be the set of graphs $G$ in $\mathcal{G}_n$ such that in every Zykov tree for $G$ whenever we start at the root and make $\ell(n)$ left turns and $r(n)$ right turns we do not encounter any node $H$ with $\omega(H) \geq \alpha\, b(n)$. Then

$$B_n \cap B_n(\ell,r) \subseteq T_n^\alpha(\ell,r) . \tag{5.6}$$

By Lemma 5.2 $\text{Prob}(B_n) \to 1$ as $n \to \infty$. Hence if Prob $B_n(\ell,r) \to 1$ as $n \to \infty$ then so does Prob $T_n^\alpha(\ell,r)$. Thus we wish to choose $\ell$ and $r$ so that Prob $B_n(\ell,r) \to 1$ as $n \to \infty$ and $\binom{\ell+r}{r}$ is as large as possible.

We now look at the complement $\bar{B}_n(\ell,r)$ of $B_n(\ell,r)$ in $\mathcal{S}_n$. Let $\mathcal{R}$ be the collection of all families $Q = (S_1,\ldots,S_b)$ of $b$ disjoint subsets of $\{1,\ldots,n\}$ with union containing $r+b$ elements. For each family $Q$ in $\mathcal{R}$ let $T_Q$ be the set of graphs $G$ in $\mathcal{S}_n$ such that the contracted graph $G_Q$ misses at most $\ell$ edges. Now if $G$ is a graph in $\bar{B}_n(\ell,r)$ then some graph obtained from $G$ by performing at most $r$ vertex-contractions contains a subgraph on $b$ vertices missing at most $\ell$ edges; and so $G \in T_Q$ for family $Q$ (proper for $G$) in $\mathcal{R}$. Hence

$$\bar{B}_n(\ell,r) \subseteq \cup \{T_Q : Q \in \mathcal{R}\} \quad . \tag{5.7}$$

Next we find an upper bound for $\text{Prob}(T_Q)$. It is convenient to let $m = \binom{b}{2}$ and $x = \frac{r+b}{b}$. We shall choose $r$ so that $x(n) \to \infty$ as $n \to \infty$. Let $N$ be a binomial random variable with parameters $m$ and $q^{x^2}$. By Lemma 3.2 for each $Q$ in $\mathcal{R}$,

$$\text{Prob}(T_Q) \leq \text{Prob}\{N \leq \ell\} \quad . \tag{5.8}$$

Now clearly $\mathcal{R}$ contains at most $n^n$ families $Q$. Hence by (5.7) and (5.8)

$$\text{Prob } \bar{B}_n(\ell,r) \leq n^n \text{ Prob}\{N \leq \ell\} \quad . \tag{5.9}$$

We shall use (5.9) to ensure that $\text{Prob } \bar{B}_n(\ell,r) \to 0$ as $n \to \infty$, and so of course we need $\ell(n) < E[N]$ (at least for large $n$).

We set

$$\ell(n) = \left\lfloor \frac{1}{2} E[N] \right\rfloor = \left\lfloor \frac{1}{2} mq^{x^2} \right\rfloor \quad . \tag{5.10}$$

Now

$$\text{Prob}\{N \leq \ell\} = \sum_{i=0}^{\ell} \binom{m}{i} (q^{x^2})^i (1-q^{x^2})^{m-i}$$

$$\leq (\ell+1)\binom{m}{\ell} q^{x^2 \ell} (1-q^{x^2})^{m-\ell} \quad . \tag{5.11}$$

Note that the right hand side above depends only on $x$ (and $n$). We have

$$\log \text{Prob}\{N \leq \ell\}$$

$$\leq \ell \log m - \ell \log \ell + \ell \log e - x^2 \ell - (m-\ell) \log e\, q^{x^2} + O(\log n)$$

$$= \ell(\log m - (\log \tfrac{1}{2} + \log m - x^2) + \log e - x^2 - 2 \log e + o(1))$$

$$= \ell(\log 2 - \log e + o(1)) \quad . \tag{5.12}$$

Now suppose that $r(n) = \lfloor \lambda n(\log n)^{-1/2} \rfloor$ for some constant $\lambda$ with $0 < \lambda < \frac{1}{2}\alpha$ say. Then $x(n) \sim (2\lambda/\alpha)(\log n)^{1/2}$ and

$$\log \ell(n) = (2 - 4\lambda^2/\alpha^2 + o(1)) \log n \quad . \tag{5.13}$$

But now by (5.9), (5.12) and (5.13)

$$\text{Prob } \bar{B}_n(\ell,r) \to 0 \quad \text{as } n \to \infty \ .$$

We next look at the value of $\binom{\ell+r}{r}$ and choose a value for $\lambda$. Now

$$\log\binom{\ell+r}{r} = r\{\log \ell - \log r + O(1)\}$$

$$= \lambda n(\log n)^{-1/2}\{2 \log n - (4\lambda^2/\alpha^2)(\log n) - \log n + O(\log \log n)\}$$

$$= (\lambda - 4\lambda^3/\alpha^2 + o(1))\, n(\log n)^{1/2} \quad .$$

The maximum value of $\lambda - 4\lambda^3/\alpha^2$ for $\lambda > 0$ is attained at $\lambda = 12^{-1/2}\alpha < \frac{1}{2}\alpha$. Thus we give $\lambda$ this value, and find that

$$\log\binom{\ell+r}{r} = (3^{-3/2} + o(1))\,\alpha n(\log n)^{1/2} \quad , \tag{5.14}$$

as required. The value we have chosen for $r$ is as in (5.5). Clearly we may decrease the value of $\ell$ from that in (5.10) if we do not thus falsify (5.14). Thus we may set $\ell$ as in (5.4). This completes the proof of Lemma 5.4. $\square$

From Lemma 5.4 and the discussion preceding it we may now deduce immediately our main results.

Theorem 5.5. If $\alpha$ is a positive constant then for almost all graphs $G_n$ in $\mathcal{G}_n$, every Zykov tree for $G_n$ is such that the logarithm of the number of nodes $H$ with $\omega(H) < \alpha\,\chi(G_n)$ is asymptotically at least

$$\alpha\, n\left(\frac{1}{27}\log n\right)^{1/2} .$$

The most interesting special case of Theorem 5.5 above is when $p = q = 1/2$ and $\alpha = 1$.

Corollary 5.6. Consider the property for graphs $G_n$ on $n$ vertices that every pruned Zykov tree for $G_n$ has size at least

$$(1.14)^{n(\log_2 n)^{1/2}} .$$

*The proportion of graphs on* $n$ *vertices with this property tends to* 1 as $n \to \infty$.

Corollary 5.6 shows that any Zykov algorithm as defined in Section 2 'almost always' requires more than exponential time. Thus certainly there exists a sequence $(G_1, G_2, \ldots, G_n, \ldots)$ such that $G_n$ is a graph on $n$ vertices and the time taken by any Zykov algorithm on $G_n$ grows faster than exponentially with $n$. No construction is known for such a sequence.

M. R. Garey and D. S. Johnston [7] have shown that the problem of determining the chromatic number of a graph to within a factor less than 2 is NP-complete. By analogy one might have expected some effect in Theorem 5.5 at $\alpha = 1/2$ say, but none is apparent (see also Corollary 7.2 below).

The above discussion is asymptotic in nature, but we may be interested in applying a Zykov algorithm to graphs which are fairly large but definitely finite, say to graphs with 500 vertices. Arguments similar to those above but simpler show that we are already in trouble. We shall see below that for more than 3/4 of the graphs on 500 vertices every pruned Zykov tree has more than $10^{12}$ nodes.

Set $p = q = 1/2$ so that probabilities correspond to proportions. We shall be talking about graphs in $\mathcal{G}_{500}$. Note first that, as in the proof of Lemma 5.2, we have

$$\begin{aligned} \text{Prob}\{\chi(G) < 39\} &\leq \text{Prob}\{\alpha(G) \geq 14\} \\ &\leq \binom{500}{14} 2^{-\binom{14}{2}} \\ &< 0.24 \quad . \end{aligned} \tag{5.15}$$

For positive integers $\ell$ and $m$ let $S(\ell,m)$ be the set of graphs $G$ in $\mathcal{G}_{500}$ which have a subgraph on $m$ vertices missing at most $\ell$ edges. Denote $\binom{m}{2}$ by $k$ and suppose that $\ell \leq \frac{1}{2} k$. Then

$$\begin{aligned} \text{Prob } S(\ell,m) &\leq \binom{500}{m} 2^{-k} \sum_{i=k-\ell}^{k} \binom{k}{i} \\ &\leq \binom{500}{m} 2^{-k} \binom{k}{\ell} \frac{k-s+1}{k-2s+1} \quad . \end{aligned}$$

It is easy to check using the above that for example

$$\text{Prob } S(53,28) < .01 \quad . \tag{5.16}$$

Let $A$ be the set of graphs $G$ in $\mathcal{G}_{500}$ such that $\chi(G) \geq 39$ and $G$ is not in $S(53,28)$. Then by (5.15) and (5.16)

$$\text{Prob } A > 0.75 \quad . \tag{5.17}$$

Now let $G$ be a graph in $A$ and let $T$ be a pruned Zykov tree for $G$. Then in $T$ if we start at the root and make $53$ left turns and $11$ right turns we can never reach a leaf; for if $H$ is a leaf of $T$ then $H$ has a complete subgraph on $39$ vertices and at least $39 - 11 = 28$ of them must be original vertices of $G$. Hence the number of leaves of $T$ is more than

$$2\binom{53+11}{11} > 5 \times 10^{11}$$

and so the number of nodes in $T$ is more than $10^{12}$. Hence by (5.17) for more than $3/4$ of the graphs $G$ in $\mathcal{A}_{500}$ every pruned Zykov tree for $G$ has more than $10^{12}$ nodes.

The basic result in this section is of course Lemma 5.4 from which Theorem 5.5 and Corollary 5.6 follow immediately. We remarked earlier that Lemma 5.4 corresponds to Lemma 4.2 and we noted in Section 4 that Lemma 4.2 is in a sense best possible. We now investigate how good Lemma 5.4 is. Proposition 5.7 below shows that in a (weaker) sense Lemma 5.4 is also best possible. This suggests that our lower bound for the size of a smallest pruned Zykov tree for a graph may not be too bad. However, our only upper bound for the size of a smallest pruned Zykov tree for a graph is very much larger (see Corollary 6.2 in the next section).

<u>Proposition 5.7</u>. Let $\alpha$ be a positive constant. If $\ell$ and $r$ are functions from $\mathbb{N}$ to $\mathbb{N}$ such that

$$\log\binom{\ell+r}{r} > (2+o(1))\,\alpha n\left(\frac{1}{27}\log n\right)^{1/2} \tag{5.18}$$

then

$$\text{Prob } T_n^{\alpha}(\ell,r) \to 0 \quad \text{as } n \to \infty . \tag{5.19}$$

Further if the Conjecture 5.3 holds and if

$$\log\binom{\ell+r}{r} > (1+o(1))\alpha n\left(\frac{1}{27}\log n\right)^{1/2} \tag{5.20}$$

then again (5.19) holds.

<u>Proof</u>. For each $n$ in $\mathbb{N}$ let $\delta(n)$ be a real number such that say $1 \le \delta(n) \le 3$. Suppose that $\ell$ and $r$ are functions from $\mathbb{N}$ to $\mathbb{N}$ such that

$$\log\binom{\ell+r}{r} > (\delta+o(1))\alpha n\left(\frac{1}{27}\log n\right)^{1/2} . \tag{5.21}$$

For each $n$ in $\mathbb{N}$ let $d(n) = (\delta/2)\, n/\log n$, let $D_n$ be the set of graphs $G_n$ in $\mathcal{G}_n$ such that $\chi(G_n) \le d(n)$, and let $D_n(\ell,r)$ be the set of graphs $G_n$ in $\mathcal{G}_n$ such that in every Zykov tree for $G_n$ whenever we start at the root and make $\ell(n)$ left turns and $r(n)$ right turns we do not encounter any node $H$ with $\omega(H) \ge \alpha\, d(n)$. Then

$$T_n^{\alpha}(\ell,r) \cap D_n \subseteq D_n(\ell,r) \quad . \tag{5.22}$$

We shall prove that

$$\text{Prob } D_n(\ell,r) \to 0 \quad \text{as} \quad n \to \infty \quad . \tag{5.23}$$

Once we have done this we are nearly finished.

Note first that we may assume that $\ell(n) \le \binom{n}{2}$ and $r(n) \le n-1$. Also if $\ell \le r$ for some $n$ in $\mathbb{N}$ then

$$\binom{\ell+r}{r} \le \binom{2r}{r} \le 2^{2n}$$

and so by (5.21) we have $\log(\ell+r) = \log \ell + O(1)$.

Now

$$\binom{\ell+r}{r} \le (\ell+r)^r \le n^{2r}$$

and so by (5.21) again

$$r(n) \ge (c_1+o(1))\, n\, (\log n)^{-1/2} \quad \text{for some constant } c_1 > 0 \quad . \tag{5.24}$$

We next show that we may assume that

$$r(n) \leq (c_2 + o(1))\, n\,(\log n)^{-1/2} \quad \text{for some constant } c_2 > 0 \ . \tag{5.25}$$

For each $n$ in $\mathbb{N}$ let $s(n) = \lceil (2 \log n)^{1/2} \rceil$ and $t(n) = \lceil \alpha d(n) \rceil$.

Then by Lemma 3.5

$$\text{Prob}\{G'_n \text{ complete}\} \to 1 \quad \text{as } n \to \infty \ . \tag{5.26}$$

But we may obtain the graph $G'_n$ from the graph $G_n$ by performing at most $(s(n)-1)t(n)$ vertex-contractions, and so

$$D_n(0, st) \subseteq \{G'_n \text{ not complete}\} \ . \tag{5.27}$$

Now by (5.26) and (5.27)

$$\text{Prob } D_n(0, st) \to 0 \quad \text{as } n \to \infty \ .$$

It follows that we may assume that (5.25) holds.

We now show that for $n$ sufficiently large we have .

$$\ell(n) \geq n^2 \, q^{(r/\alpha d - 1)^2} \ . \tag{5.28}$$

Let

$$x(n) = r(n)(\log n)^{1/2}/n$$

so that by (5.24) and (5.25) we have $\log x = O(1)$ . Note that

$$r/\alpha d = 2x/\alpha\delta \ (\log n)^{1/2} \ .$$

Now if (5.28) is false then for infinitely many values of $n$ we have

$$\ell(n) < n^2 q^{(r/\alpha d - 1)^2}$$

and so

$$\begin{aligned} \log\binom{\ell+r}{r} &= r(\log \ell - \log r + O(1)) \\ &\leq x\, n(\log n)^{-1/2}(2 \log n - r^2/\alpha^2 d^2 + 2r/\alpha d - \log n \\ &\qquad + \tfrac{1}{2} \log \log n + O(1)) \\ &= (x - 4x^3/\alpha^2\delta^2 + o(1))\, n\,(\log n)^{1/2} \\ &\leq (\delta + o(1))\, \alpha n (\tfrac{1}{27} \log n)^{1/2} \end{aligned}$$

(see the proof of Lemma 5.4). But this contradicts (5.21) and so (5.28) must hold.

Now for each $n$ in $\mathbb{N}$ let $s(n) = \lceil r(n)/\alpha d(n) \rceil - 1$ and $t(n) = \lceil \alpha d(n) \rceil$. By (5.25) and Lemma 3.5

$$\mathrm{Prob}\{Q(G_n) \text{ full}\} \to 1 \quad \text{as } n \to \infty \ . \tag{5.29}$$

Also

$$\begin{aligned}(s(n)-1)t(n) &\le (r/\alpha d - 1)(\alpha d + 1) \\ &\le r\end{aligned}$$

for $n$ sufficiently large that $\alpha^2 d^2 \ge r$. Hence as in the derivation of (5.27) we have that for $n$ sufficiently large

$$D_n(\ell, r) \subseteq \{G_n' \text{ misses more than } \ell \text{ edges}\} \ . \tag{5.30}$$

For each $n$ in $\mathbb{N}$ let $N$ be a binomial random variable with parameters $\binom{t}{2}$ and $q^{s^2}$. Then by Lemma 3.3

$$\begin{aligned}&\mathrm{Prob}\{G_n' \text{ misses more than } \ell \text{ edges}\} \\ &\quad \le 1 - \mathrm{Prob}\{N \le \ell\}\ \mathrm{Prob}\{Q(G_n) \text{ full}\} \ .\end{aligned} \tag{5.31}$$

But $\ell(n) \to \infty$ as $n \to \infty$ and by (5.28) $\ell(n)/E[N] \to \infty$ as $n \to \infty$. Hence

$$\mathrm{Prob}\{N \le \ell\} \to 1 \quad \text{as } n \to \infty \ . \tag{5.32}$$

But now (5.23) follows by (5.29), (5.30), (5.31) and (5.32).

Suppose that $\kappa(n) = 2 + \varepsilon(n)$ for $n$ in $\mathbb{N}$, where $\varepsilon(n) > 0$ and $\varepsilon(n) \to 0$ as $n \to \infty$ sufficiently slowly that by Theorem 8 in [8] we have

$$\mathrm{Prob}\ D_n \to 1 \quad \text{as } n \to \infty \ . \tag{5.33}$$

Then (5.19) follows from (5.21), (5.22) and (5.33) and so we have proved that if (5.18) is true then so is (5.19). Now suppose that the Conjecture 5.3 is true and that $\delta(n) = 1 + \varepsilon(n)$ for $n$ in $\mathbb{N}$, where $\varepsilon(n) > 0$ and $\varepsilon(n) \to 0$ as $n \to \infty$ sufficiently slowly that (5.33) holds. Then as above it follows that if the Conjecture 5.3 and (5.20) are true then so is (5.19). This completes the proof of Proposition 5.7. □

## 6. Backtrack Coloring.

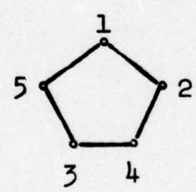
In this section we investigate the 'backtrack' coloring algorithm (BC algorithm) for determining the chromatic number of a graph. This algorithm was pointed out to the author by R. Tarjan. Given a graph $G$ it explores part of the 'backtrack coloring tree' (BC tree) for $G$, which is an implicit enumeration of the proper partitions of $G$. We shall see that the BC algorithm is essentially the same as a certain Zykov algorithm, the 'marked' Zykov algorithm. Also we shall give an upper bound for the number of nodes of the BC tree explored by the BC algorithm. It will follow that it is worth pruning BC and Zykov trees.

We first describe the backtrack coloring tree (BC tree) for a graph $G$ in $\mathcal{G}_n$. It is a rooted tree with height $n-1$. Each node is colored with one of the colors $c_1,\ldots,c_n$. A node colored $c_i$ at depth $d$ (distance $d$ below the root) corresponds to an assignment of color $c_i$ to vertex $(d+1)$ of $G$. By looking at a node and its ancestors we see that a node at depth $d$ corresponds to a coloring of the first $(d+1)$ vertices of $G$. To construct the BC tree for $G$ we first construct a single node (the root) and color it $c_1$. Now suppose that $K$ is a leaf in the tree so far constructed and that $K$ is at depth $d \leq n-2$. Then $K$ corresponds to a proper coloring $C$ of the first $(d+1)$ vertices of $G$. Let $i_0$ be $1$ plus the maximum index of a color used in the coloring $C$; and let $c_{i_1},\ldots,c_{i_j}$ (where $j \geq 0$ and $i_1 > \ldots > i_j$) be the colors used in the coloring $C$ and such that vertex $(d+2)$ is not adjacent to any vertex of the color. We let the node $K$ have $(j+1)$ sons colored $c_{i_0},c_{i_1},\ldots,c_{i_j}$ in order from left to right.

We have now defined the BC tree for $G$. It is not hard to see that there is a 1-1 correspondence between the nodes of the BC tree for $G$ at depth $d$ and the proper partitions of the subgraph of $G$ induced by the first $(d+1)$ vertices (see Example 6.1 below). Hence the number of nodes in the BC tree for $G$ is between $C(G)$ and $nC(G)$, and so Theorem 4.3 gives asymptotic results about the size of BC trees.

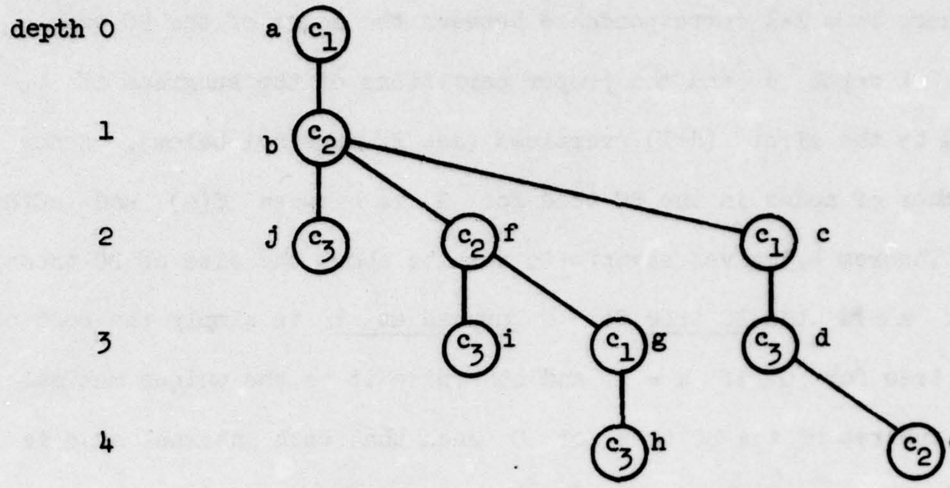
If $k \in \mathbb{N}$ the BC tree for $G$ pruned at $k$ is simply the root of the BC tree for $G$ if $k = 1$ and otherwise it is the unique maximal rooted subtree of the BC tree for $G$ such that each internal node is colored with one of the first $(k-1)$ colors. The pruned BC tree for $G$ is the BC tree for $G$ pruned at $\chi(G)$.

Example 6.1. Take $G$ as the cycle with 5 vertices, numbered as indicated.

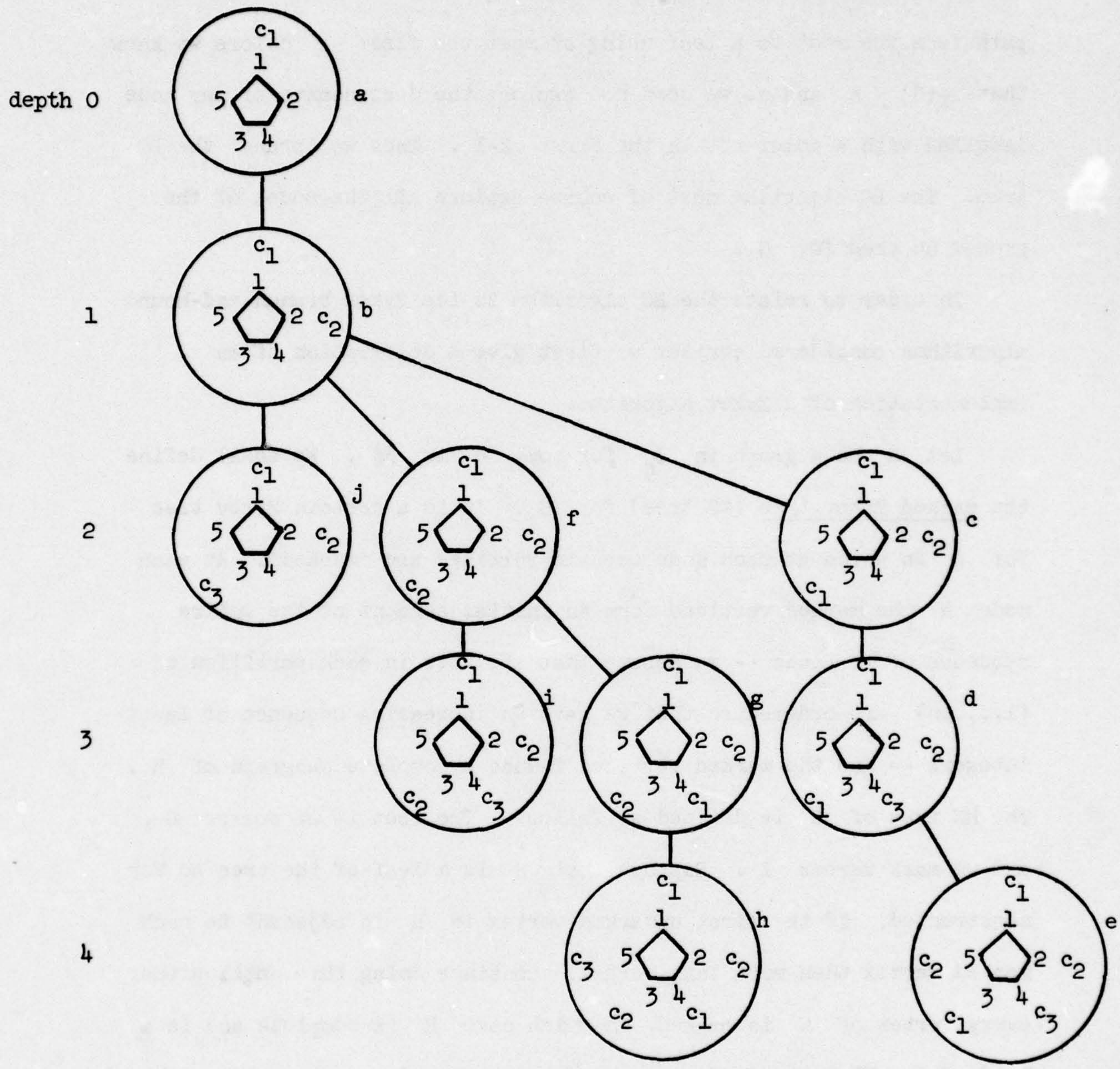



In (a) below we show the part of the BC tree for $G$ explored by the BC algorithm. In (b) we show the same tree structure and indicate at each node the corresponding partial coloring of $G$. The letters a,...,j indicate the order in which the nodes are first visited by the BC algorithm.

(a)



(b)



The backtrack coloring algorithm (BC algorithm) for determining the chromatic number $\chi(G)$ of a graph $G$ conducts a depth-first search of the BC tree for $G$, keeping to the right. Once we have found a path from the root to a leaf using at most the first $k$ colors we know that $\chi(G) \leq k$ and so we need not explore the descendants of any node labelled with a color not in the first $k-1$. Thus we 'prune' the BC tree. The BC algorithm must of course explore all the nodes of the pruned BC tree for $G$.

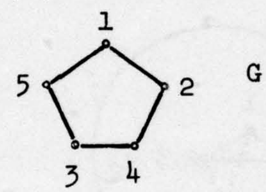
In order to relate the BC algorithm to the Zykov branch-and-bound algorithms considered earlier we first give a description of an implementation of a Zykov algorithm.

Let $G$ be a graph in $\mathcal{J}_n$ for some $n$ in $\mathbb{N}$. We shall define the marked Zykov tree (MZ tree) for $G$. It is a certain Zykov tree for $G$ in which at each node certain vertices are 'marked'. At each node $H$ the marked vertices form an initial segment of the entire sequence of vertices -- we assume that the sets in each partition of $\{1,\dots,n\}$ are ordered so that we have an increasing sequence of least integers -- and the marked vertices induce a complete subgraph of $H$. The MZ tree of $G$ is defined as follows. The root is of course $G$, and we mark vertex $1$. Suppose that $H$ is a leaf of the tree so far constructed. If the first unmarked vertex in $H$ is adjacent to each marked vertex then mark this vertex. Continue doing this until either every vertex of $H$ is marked, in which case $H$ is complete and is a leaf of the MZ tree of $G$; or the first unmarked vertex is not adjacent in $H$ to some marked vertex. In this case we branch on the first unmarked vertex and the first marked vertex not adjacent to it. Marked

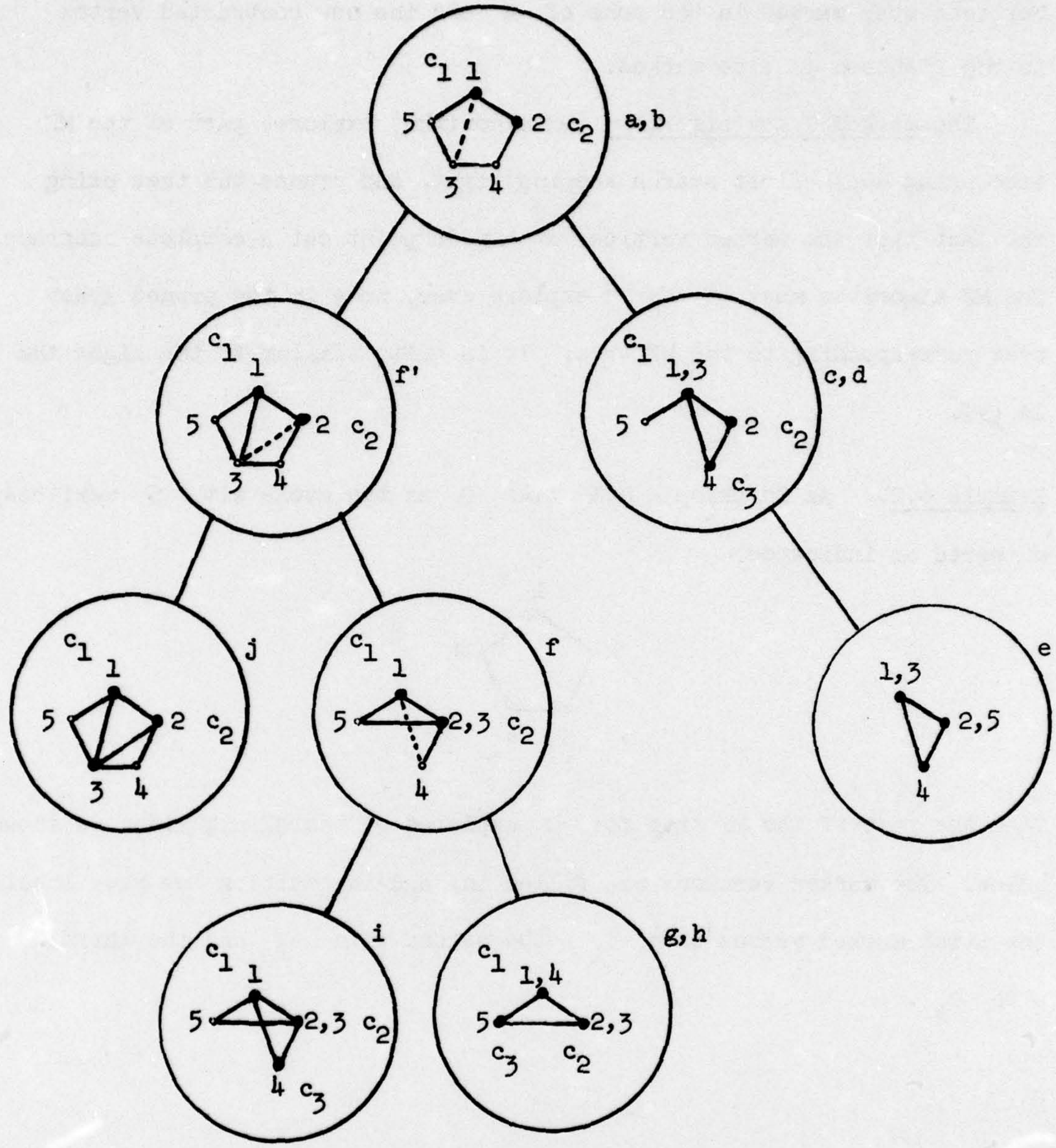
vertices stay marked in the sons of H and the new contracted vertex in the rightson is also marked.

The <u>marked Zykov algorithm</u> (MZ algorithm) explores part of the MZ tree using depth-first search keeping right, and prunes the tree using the fact that the marked vertices at a node point out a complete subgraph. The MZ algorithm must of course explore every node in the pruned Zykov tree corresponding to the MZ tree. It is quite similar to the algorithm in [5].

<u>Example 6.2</u>. As in Example 6.1 take G as the cycle with 5 vertices, numbered as indicated.




Then the part of the MZ tree for G explored by the MZ algorithm is shown below. The marked vertices are filled in, and in addition we have labelled the first marked vertex with $c_1$, the second with $c_2$ and the third with $c_3$.


c1 1
5 2 c2
3 4
a,b
c1 1
5 2 c2
3 4
f'
c1 1,3
5 2 c2
4 c3
c,d
c1 1
5 2 c2
3 4
j
c1 1
5 2,3 c2
4
f
1,3
2,5
4
e
c1 1
5 2,3 c2
4 c3
i
c1 1,4
5 2,3
c3 c2
g,h

It should be apparent that the BC and MZ algorithms are really different forms of the same algorithm. Suppose that $G$ is a graph in $\mathcal{G}_n$ . Then it is not hard to prove that there is a correspondence between the nodes of the BC tree $B$ for $G$ and the nodes of the MZ tree $Z$ for $G$ such that

(a) each node in $B$ corresponds to one or two nodes in $Z$ ;

(b) each node in $Z$ corresponds to between $1$ and $n$ nodes in $B$ ;

(c) pruning occurs at corresponding nodes.

The lettering in Examples 6.1 and 6.2 indicates such a correspondence. Let $B^*$ and $Z^*$ be the parts of the trees $B$ and $Z$ explored by the BC and MZ algorithms respectively. Then by the above

$$2|B^*| \geq |Z^*| \quad \text{and} \quad n|Z^*| \geq |B^*| \quad .$$

It follows by Corollary 5.6 that for almost all graphs on $n$ vertices the BC algorithm requires time at least $c^{n(\log n)^{1/2}}$ for some constant $c > 1$ . The next result yields an upper bound for the time required by the BC or MZ algorithm.

<u>Theorem 6.1</u>. Let $\varepsilon > 0$ . Then for almost all graphs in $\mathcal{G}_n$ the number of nodes of the BC tree explored by the BC algorithm is at most $n^{(\frac{1}{2}+\varepsilon)n}$ . If Conjecture 5.3 is true then for almost all graphs $G_n$ in $\mathcal{G}_n$ the pruned BC tree for $G_n$ has at most $n^{(\frac{1}{4}+\varepsilon)n}$ nodes.

<u>Proof</u>. Let $k$ be a function from $\mathbb{N}$ to $\mathbb{Z}$ . For each graph $G$ in $\mathcal{G}_n$ let $B^k(G)$ be the BC tree for $G$ pruned at $k$ . For $i,j$ in $\mathbb{N}$ let $f(i,j)$ be the expected number of proper partitions into $j$ sets of graphs in $\mathcal{G}_i$ . Then

$$E[|B^k(G_n)|] \leq \sum_{i=1}^{n} \sum_{j=1}^{k} f(i,j) \quad . \tag{6.1}$$

From the proof of Theorem 3.3 we have

$$f(i,j) \leq j^i \, q^{\frac{1}{2}(i^2/j - i)}$$

and so if $i \leq n$ and $j \leq k$ we certainly have

$$f(i,j) \leq n^i \, q^{i^2/2k} \, q^{-\frac{1}{2}n} \quad . \tag{6.2}$$

Now let

$$k(n) = \lfloor (1+\varepsilon)n/\log n \rfloor$$

for $n$ in $\mathbb{N}$ . Then for $i \in \mathbb{N}$ , $i \leq n$

$$n^i \, q^{i^2/2k} \leq n^n \, q^{n^2/2k}$$

and so by (6.1) and (6.2)

$$E[|B^k(G_n)|] \leq n^{n+2} \, q^{n^2/2k} \, q^{-\frac{1}{2}n}$$

$$\leq n^{(\frac{1}{2} + \frac{\varepsilon}{2} + o(1))n} \quad .$$

Hence

$$\mathrm{Prob}\{|B^k(G_n)| \leq n^{(\frac{1}{2} + \varepsilon)n}\} \to 1 \quad \text{as } n \to \infty \quad . \tag{6.3}$$

Now the BC algorithm initially explores the 'rightmost' path in the BC tree, and so initially it acts like the sequential coloring algorithm. Hence by Theorem 8 in [8], for almost all graphs in $\mathcal{G}_n$ the BC algorithm explores at most $n$ nodes of the BC tree which are not in the BC tree pruned at $k$ . The first part of Theorem 6.1 now follows from (6.3).

We now prove the second part of the theorem. Let

$$k(n) = \lfloor (1+\varepsilon) \tfrac{1}{2} n/\log n \rfloor$$

for $n$ in $\mathbb{N}$. Then for $i \in \mathbb{N}$, $i \leq n$

$$n^i q^{i^2/2k} \leq n^{\frac{1}{4}(1+\varepsilon)n}$$

and so by (6.1) and (6.2)

$$E[|B^k(G_n)|] \leq n^{(\frac{1}{4}(1+\varepsilon)+o(1))n} .$$

Hence as above

$$\text{Prob}\{|B^k(G_n)| \leq n^{(\frac{1}{4}+\varepsilon)n}\} \to 1 \quad \text{as} \quad n \to \infty . \tag{6.4}$$

Denote the pruned BC tree for a graph $G$ by $B^*(G)$. If $\chi(G) \leq k$ then $|B^*(G)| \leq |B^k(G)|$. Thus

$$\{|B^*(G_n)| \leq n^{(\frac{1}{4}+\varepsilon)n}\} \supseteq \{|B^k(G_n)| \leq n^{(\frac{1}{4}+\varepsilon)n}\} \cap \{\chi(G_n) \leq k\} . \tag{6.5}$$

Now suppose that Conjecture 5.3 holds, so that

$$\text{Prob}\{\chi(G_n) \leq k\} \to 1 \quad \text{as} \quad n \to \infty . \tag{6.6}$$

Then the second part of Theorem 6.1 follows from (6.4), (6.5) and (6.6). □

<u>Corollary 6.2</u>. Let $\varepsilon > 0$. Then for almost all graphs $G_n$ in $\mathcal{G}_n$ the number of nodes of the marked Zykov tree for $G_n$ explored by the marked Zykov algorithm is at most $n^{(\frac{1}{2}+\varepsilon)n}$. If Conjecture 5.3 holds then for almost all graphs $G_n$ in $\mathcal{G}_n$ the pruned marked Zykov tree for $G_n$ has at most $n^{(\frac{1}{4}+\varepsilon)n}$ nodes.

## 7. Lengths of Proofs.

Most of our results so far may be phrased in terms of the lengths of certain kinds of proof which determine chromatic numbers or which establish lower bounds for chromatic numbers. We then obtain results concerning chromatic numbers which are similar in spirit to recent results of V. Chvatal [4] concerning stability numbers. Indeed this paper was initially motivated by discussions with Chvatal concerning his results.

If $k$ is an integer at least as great as $\chi(G)$ then there is a short proof that $\chi(G) \leq k$ -- namely we may exhibit a coloring of $G$ using at most $k$ colors. In general such a proof is hard to find but it must of course exist. However, if $k$ is at most $\chi(G)$ then it is not clear if there is necessarily a short proof of this fact.

The following two rules may be used to determine or bound chromatic numbers (see Section 2 and (2.1) in particular).

(R1) $\chi(G) = \min\{\chi(G'_{xy}), \chi(G''_{xy})\}$ .

(R2) If $G$ is complete then $\chi(G)$ equals the number of vertices of $G$ .

Given a set $S$ of rules like (R1) and (R2) let us call a proof that uses only these rules an <u>S-proof</u>, and each application of a rule in $S$ a <u>step</u>. Clearly there is a close correspondence between an {(R1),(R2)}-proof determining $\chi(G)$ and a Zykov tree for $G$ .

From Theorem 4.1 we obtain

<u>Corollary 7.1</u>. If $G$ is a graph in $\mathcal{S}_n$ then every {(R1),(R2)}-proof which determines $\chi(G)$ without redundant steps has exactly $2C(G)-1$ steps.

Thus by Theorem 4.3 we know quite a lot about the lengths of {(R1),(R2)}-proofs which determine chromatic numbers. Consider now a

third rule, which can be used to establish a lower bound for chromatic numbers (see (2.2)).

(R3) $\chi(G) \geq \omega(G)$ .

Allowing the use also of the rule (R3) corresponds to pruning our Zykov trees. From Theorem 5.5 we obtain

<u>Corollary 7.2</u>. If $\alpha$ is a given constant factor with $0 < \alpha \leq 1$ then for almost all graphs $G_n$ in $\mathcal{G}_n$ every {(R1),(R3)}-proof which establishes a lower bound for $\chi(G_n)$ exact to within the factor $\alpha$ is such that the logarithm of the number of steps is asymptotically at least

$$\alpha\, n\left(\frac{1}{27}\log n\right)^{1/2} .$$

Now set $p = q = 1/2$ and $\alpha = 1$ in Corollary 7.2 (as we did in Theorem 5.5).

<u>Corollary 7.3</u>. Consider the property for graphs $G_n$ in $\mathcal{G}_n$ that in every {(R1,(R3)}-proof establishing the correct lower bound for $\chi(G_n)$ the number of steps is at least

$$(1.14)^{n(\log_2 n)^{1/2}} .$$

The proportion of graphs in $\mathcal{G}_n$ with this property tends to 1 as $n \to \infty$ .

From Corollary 6.2 we obtain

<u>Corollary 7.4</u>. Let $\varepsilon > 0$ . Then for almost all graphs $G_n$ in $\mathcal{G}_n$ the marked Zykov algorithm yields and {(R1),(R2),(R3)}-proof determining $\chi(G_n)$ with at most $n^{(\frac{1}{2}+\varepsilon)n}$ steps. If Conjecture 5.3 holds then for almost all graphs $G_n$ in $\mathcal{G}_n$ the marked Zykov algorithm (eventually)

yields an $\{(R1),(R2),(R3)\}$-proof determining $\chi(G_n)$ with at most $n^{(\frac{1}{4}+\varepsilon)n}$ steps.

Consider now a fourth rule which can be used to bound chromatic numbers.

(R4) If $G$ has a subgraph $H$ then $\chi(G) \geq \chi(H)$ .

The set of rules $\{(R1),(R2),(R4)\}$ seems to the author to be as natural as the set $\{(R1),(R3)\}$ for establishing lower bounds for chromatic numbers. The following proposition shows that the two sets of rules are in a sense equivalent. The proof is straightforward and is omitted.

Proposition 7.5. For any $\{(R1),(R3)\}$-proof that $\chi(G) \geq k$ there is an $\{(R1),(R2),(R4)\}$-proof with at most twice as many steps; and for any $\{(R1),(R2),(R4)\}$-proof that $\chi(G) \geq k$ there is an $\{(R1),(R3)\}$-proof with no more steps.

At first sight it might seem to be of advantage to allow also rules like the rule (R5) below, which is closely related to the rule (R1).

(R5) $\chi(G) \geq \max\{\chi(G'_{xy}), \chi(G''_{xy})\}-1$ .

One would of course not have to know both $\chi(G'_{xy})$ and $\chi(G''_{xy})$ in order to use the rule (R5). However, it is not hard to prove for example the following proposition.

Proposition 7.6. For any $\{(R1),\ldots,(R5)\}$-proof that $\chi(G) \geq k$ there is an $\{(R1),(R3)\}$-proof with no more steps.

Another rule which might be considered is the following.

(R6) If some vertex $v$ in $G$ is adjacent to each other vertex then

$$\chi(G) = \chi(G-v)+1 \quad \text{(where } G-v \text{ has the obvious meaning).}$$

However, again we may see without difficulty that including this rule would not lead to shorter proofs.

Yet another possible rule which might be thought helpful is the 'principle of separation into pieces', as described in [2] Chapter 15. This rule shows how to break our problem into smaller independent subproblems if the graph has a separating set which induces a complete subgraph. It may on occasion help to organize proofs but once again we may easily check that it does not shorten them.

Finally let us note that all the above discussion falls down if we are allowed to recognize isomorphic graphs with different vertex sets. It would be interesting to know what can be said in this case.

8. <u>Minimal Colorings</u>.

Many authors have investigated algorithms $A$ for (properly) coloring graphs $G$ which are fairly fast but which use a number $A(G)$ of colors possibly greater than $\chi(G)$. (See for example [9], [12], [13], [14].) Following D. S. Johnson [9] we let $\hat{A}(G)$ be the ratio of $A(G)$ to $\chi(G)$, and let $\hat{A}(n)$ be the maximum value of $\hat{A}(G)$ over all graphs $G$ on $n$ vertices. Clearly $1 \leq \hat{A}(n) \leq n$ and the smaller $\hat{A}(G)$ or $\hat{A}(n)$ is the better. In [9] it is shown that for several of the most common algorithms $A$ the function $\hat{A}(n)$ is of order $n$. For the best of the known (fast) algorithms the function $\hat{A}(n)$ is still of order $n/\log n$.

It is suggested in [9] that the usual behavior of $\hat{A}(G_n)$ for graphs $G_n$ on $n$ vertices may be very different from the behavior found for $\hat{A}(n)$. We shall see that this is indeed the case.

Consider first the sequential coloring algorithm SA or $A_1$ (see [8], [9] and Section 3 of this paper). Johnson shows without difficulty that $\hat{A}_1(n)$ is of order $n$, and suggests that, however, the expected value of $\hat{A}_1(G_n)$ may be bounded by a constant independent of $n$. It follows easily from results in a paper [8] by G. Grimmett and the present author that for any $\varepsilon > 0$ we have $\hat{A}_1(G_n) \leq 2+\varepsilon$ for almost all graphs $G_n$ in $\mathcal{G}_n$: also it is easy to prove that the expected value of $\hat{A}_1(G_n)$ is at most $2+\varepsilon$ for $n$ sufficiently large (see the proof of Theorem 8.2 below).

We now look at the usual behavior of $\hat{A}(G_n)$ for other coloring algorithms $A$. A proper coloring of a graph $G$ is <u>minimal</u> if for each pair of colors used some vertex of one color is adjacent to some vertex of the other color; that is, if no color can be replaced by some other already used color; that is, if the corresponding proper partition $Q$

of $G$ is such that the contracted graph $G_Q$ is complete. A coloring algorithm is <u>minimal</u> if it always yields minimal colorings. All the usual coloring algorithms are minimal, and in any case from an arbitrary proper coloring one may easily produce a minimal coloring. Thus it seems reasonable to restrict our attention to minimal coloring algorithms.

For every graph $G$ let $M(G)$ be the maximum value of $A(G)$ over all minimal coloring algorithms $A$. An alternative definition of $M(G)$ is then that it is the largest integer $t$ for which there exists a proper partition $Q$ of $G$ into $t$ sets such that the contracted graph $G_Q$ is complete. For every graph $G$ we let $\hat{M}(G)$ be the ratio of $M(G)$ to $\chi(G)$. Thus $\hat{M}(G)$ is a measure of how badly it is possible to color $G$.

It seems that for any fast coloring algorithm $A$ yet proposed there exist graphs on which $A$ performs very badly ([9]). However, for most graphs every minimal coloring algorithm performs not too badly: we shall prove below that $\hat{M}(G_n)$ is in probability only of order $(\log n)^{1/2}$.

<u>Lemma 8.1</u>. Let $\varepsilon > 0$. Then for almost all graphs $G_n$ in $\mathcal{G}_n$

$$(1-\varepsilon)\, n\, (2 \log n)^{-1/2} \leq M(G_n) \leq (1+\varepsilon)\, n\, (\log n)^{-1/2} . \tag{8.1}$$

Further for $n$ sufficiently large the expected value of $M(G_n)$ lies in the above range.

<u>Proof</u>. The left hand inequality in (8.1) follows immediately from Lemma 3.6.

Let $m$ be an integer at least $(1+\varepsilon)\, n\, (\log n)^{-1/2}$. By Lemma 3.2 the probability that a given partition $Q$ of $\{1,\ldots,n\}$ into $m$ sets yields a complete graph $G_Q$ is at most

$$(1 - q^{(n/m)^2})^{\binom{m}{2}} .$$

Hence the probability $P_m$ that there exists such a partition $Q$ (proper or not) is at most

$$n^n(1 - q^{(n/m)^2})^{\binom{m}{2}} \quad .$$

But now

$$\log P_m \le n \log n - \binom{m}{2} \log e\, q^{(n/m)^2} < -n$$

if $n$ is sufficiently large. Hence for $n$ sufficiently large

$$\mathrm{Prob}\{M(G_n) \ge (1+\varepsilon)\, n\, (\log n)^{-1/2}\} \le n\, q^n \quad . \qquad (8.2)$$

The right hand inequality in (8.1) follows from (8.2), and so we have completed the proof of (8.1).

The second part of the lemma, concerning expected values, follows from the left hand inequality in (8.1) and from (8.2). □

Recall that $\hat{M}(G)$ is the ratio of $M(G)$ to $\chi(G)$ .

Theorem 8.2. Let $\varepsilon > 0$ . Then for almost all graphs $G_n$ in $\mathcal{G}_n$

$$(2^{-1/2} - \varepsilon)(\log n)^{1/2} \le \hat{M}(G_n) \le (2+\varepsilon)(\log n)^{1/2} \quad . \qquad (8.3)$$

Further for $n$ sufficiently large the expected value of $\hat{M}(G_n)$ lies in the above range.

Proof. We know from [8] (see also [6] Chapter 11) that for almost all graphs $G_n$ in $\mathcal{G}_n$

$$1/2\; n/\log n \le \chi(G_n) \le (1+\varepsilon)\; n/\log n \quad . \qquad (8.4)$$

Now (8.3) follows from (8.4) and Lemma 8.1.

The left hand inequality for the expected value of $\hat{M}(G_n)$ follows from the left hand inequality in (8.3). For the right hand inequality note first that

$$E[\hat{M}(G_n)] \leq (1/2\ n/\log n)^{-1} E[M(G_n)]$$

$$+ n \operatorname{Prob}\{\chi(G_n) < 1/2\ n/\log n\} \quad . \tag{8.5}$$

But from the proof of Lemma 5.2

$$n \operatorname{Prob}\{\chi(G_n) < 1/2\ n/\log n\} \to 0 \quad \text{as} \quad n \to \infty \ , \tag{8.6}$$

and by Lemma 8.1 for $n$ sufficiently large

$$E[M(G_n)] \leq (1 + \varepsilon/3)\, n\, (\log n)^{-1/2} \quad . \tag{8.7}$$

Hence by (8.5), (8.6) and (8.7)

$$E[\hat{M}(G_n)] \leq (2+\varepsilon)(\log n)^{1/2}$$

for $n$ sufficiently large. This completes the proof of this the final theorem. □

Some Questions.

The main result has been that Zykov algorithms for determining the chromatic number of a graph in probability take time at least

$$c^{n(\log n)^{1/2}} \qquad \text{(for some constant } c > 1 \text{)}$$

on graphs on $n$ vertices. This result raises at least three questions that merit attention.

Firstly, the best upper bound here for the time taken is very much greater than the lower bound. Is the lower bound of the right order of magnitude?

Secondly, all the results here are based on the random graph model which has constant edge-probability $p$, and in certain circumstances the model which has constant average degree say might be more appropriate (see for example [6] Chapter 16). Are there corresponding results for this case?

Thirdly, it follows from the discussion in Section 7 that various 'improvements' in the Zykov algorithms do not in fact lead to a decrease in the time taken. But what happens if say we allow an isomorphism search?

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